1 Analysis of Functions

Let \( d_i : (0, 1) \to \{0, 1\} \) where \( d_i(x) \) is the \( i \)-th digit of \( x \) in base 2, writing always the developments with an infinite number of 1 to remove ambiguity. Define \( r_i(x) = 2d_i(x) - 1 \) (Rademacher’s function) and \( s_n(x) = \sum_{i=1}^{n} r_i(x) \). Denote \( \mu \) the Lebesgue measure.

1. State the definition of simple functions, and prove that they are dense in \( L^1(\mathbb{R}) \).
2. State and prove Chebychev’s inequality.
3. Prove that \( \int_0^1 s_n(x) \, dx = 0 \), that \( \int_0^1 r_i(x) r_j(x) \, dx = 0 \) for \( i \neq j \) and \( \int_0^1 (s_n)^2 \, dx = n \).
4. Prove that \( \lim_{n \to \infty} \mu(\{x \in (0, 1] \mid |(\frac{1}{\pi} \sum_{i=1}^{n} d_i(x)) - \frac{1}{2}| \geq \varepsilon \}) = 0 \).
5. Prove that \( \int_0^1 (s_n(x))^4 \, dx \leq 3n^2 \) and deduce \( \mu(\{x \in (0, 1] \mid |s_n(x)| \geq n \varepsilon \}) \leq \frac{3}{n^2 \pi} \).
6. Prove (Borel’s theorem) that \( N = \{x \in (0, 1] \mid \lim_{n \to \infty} \frac{1}{n} s_n = 0\} \) has measure 1. [Hint: Choose \( \varepsilon_n \) s.t. \( \frac{1}{n \varepsilon_n} \) is summable and compare \( I \setminus N \) and \( \{x \in (0, 1] \mid |s_n(x)| \geq n \varepsilon_n\} \).]

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Integration in \( \mathbb{R} \) and \( \mathbb{R}^2 \) is done with the standard Lebesgue measure.

1. Recall the definitions of the Fourier transform \( \mathcal{F}f \) the Fourier transform of \( f \in L^1(\mathbb{R}) \), and the Fourier-Plancherel transform \( \hat{g} \) of \( g \in L^2(\mathbb{R}) \). Prove that if \( u \in L^2(\mathbb{R}) \) and \( v \in L^1(\mathbb{R}) \), the Fourier-Plancherel transform of \( u \ast v \) exists and equals \( \hat{u} \ast \mathcal{F}v \).
2. For which \( p \in [1, +\infty] \) do we have \( N \in L^p(\mathbb{R}^2) \) where \( N(x, y) := \frac{y-x}{y+x} \)?
3. Prove that \( \phi : \Delta = \{(a, b) \in \mathbb{R}^2 \mid 0 \leq a \leq b\} \) defined by \( \phi(a, b) = \int_a^b \frac{\sin t}{t} \, dt \) is continuous and bounded. [Hint: We remind the following result that can be used without proof: the improper integral \( \int_0^\infty \frac{\sin t}{t} \, dt \) exists and is \( \pi/2 \).]
4. Show that the Fourier transform \( \mathcal{F}g_k \) of \( g_k := \chi_{1/k < |x| < 1/k} \), \( k \geq 1 \), is well-defined and bounded independently of \( k \), and converges pointwise to a certain function \( g \).
5. Prove that if \( f \in L^2(\mathbb{R}) \), the convolution \( f \ast g_k \) converges in \( L^2(\mathbb{R}) \) to a function \( H(f) \in L^2(\mathbb{R}) \) (called the Hilbert transform of \( f \)).
6. Prove that \( \|H(f)\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})} \) and \( H(H(f)) = -f \).

(*) Bonus: not needed to get full mark on the question. Consider \( f \in L^1(\mathbb{R}) \) so that \( F_y(x) := N(x, y)f(x) \) is integrable for almost every \( y \in \mathbb{R} \). Prove that \( f \) is zero almost everywhere. [Hint: Use Lesbegue’s differentiation Theorem.]
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Recalls: A topological space is separable if it contains a countable dense subset. The dual $E'$ of a normed vector space $E$ is the space of continuous linear forms on $E$.

1. Give (without proof) a countable dense subset of $L^p(\mathbb{R})$ when $p \in [1, \infty)$.

2. Prove that $L^\infty(\mathbb{R})$ is not separable.

3. Prove that if the dual $E'$ of normed vector space $E$ is separable then $E$ itself is separable. [Hint: Use the Hahn-Banach Theorem.]

4. Prove that $L^1(\mathbb{R})$ is not the dual space of $L^\infty(\mathbb{R})$.

5. Recall what is the generalised derivative $D(f)$ of a function $f \in L^2(\mathbb{R})$.

6. For $f \in L^2(\mathbb{R})$ and $h > 0$ define $\tau_h f \in L^2(\mathbb{R})$ by $\tau_h f(x) := f(x + h)$. Assume that there is $C > 0$ s.t. $\|\tau_h f - f\|_{L^2(\mathbb{R})} \leq C|h|$ for all $h > 0$, then prove that $D(f)$ is an $L^2(\mathbb{R})$ function and $\frac{\tau_h f - f}{h}$ converges to $D(f)$ in the weak $L^2(\mathbb{R})$ topology as $h \to 0$. 