Examiner:

1 Analysis of Functions

Let $d_i : (0, 1] \to \{0, 1\}$ where $d_i(x)$ is the i-th digit of x in base 2, writing always the developments with an infinite number of 1 to remove ambiguity. Define $r_i(x) = 2d_i(x) - 1$ (Rademacher's function) and $s_n(x) = \sum_{i=1}^{n} r_i(x)$. Denote μ the Lebesgue measure.

1. State the definition of *simple functions*, and prove that they are dense in $L^1(\mathbb{R})$.
2. State and prove Chebychev's inequality.
3. Prove that $\int_0^1 s_n(x)\,\mathrm{d}x = 0$, that $\int_0^1 r_i(x)r_j(x)\,\mathrm{d}x = 0$ for $i \neq j$ and $\int_0^1 (s_n)^2\,\mathrm{d}x = n$.
4. Prove that $\lim_{n\to\infty} \mu(\{x \in (0, 1] \mid |(\frac{1}{n}\sum_{i=1}^{n} d_i(x)) - \frac{1}{2}| \geqslant \varepsilon\}) = 0$.
5. Prove that $\int_0^1 (s_n(x))^4\,\mathrm{d}x \leqslant 3n^2$ and deduce $\mu(\{x \in (0, 1] \mid |s_n(x)| \geqslant n\varepsilon\}) \leqslant \frac{3}{n^2\varepsilon^4}$.
6. Prove (Borel's theorem) that $N = \{x \in (0, 1] \mid \lim_{n\to\infty} \frac{1}{n}s_n = 0\}$ has measure 1. [*Hint: Choose* ε_n *s.t.* $\frac{1}{n^2\varepsilon_n^4}$ *is summable and compare* $I \setminus N$ *and* $\cup\{x \in (0, 1] \mid |s_n(x)| \geqslant n\varepsilon_n\}$.]

1 Analysis of Functions

Integration in $\mathbb{R}$ and $\mathbb{R}^2$ is done with the standard Lebesgue measure.

1. Recall the definitions of the Fourier transform $\mathcal{F}f$ the Fourier transform of $f \in L^1(\mathbb{R})$, and the Fourier-Plancherel transform $\hat{g}$ of $g \in L^2(\mathbb{R})$. Prove that if $u \in L^2(\mathbb{R})$ and $v \in L^1(\mathbb{R})$, the Fourier-Plancherel transform of $u * v$ exists and equals $\hat{u} \cdot \mathcal{F}v$.
2. For which $p \in [1, +\infty]$ do we have $N \in L^p(\mathbb{R}^2)$ where $N(x, y) := \frac{\chi_{x\neq y}}{y-x}$?
3. Prove that $\phi : \Delta = \{(a, b) \in \mathbb{R}^2 \mid 0 \leqslant a \leqslant b\}$ defined by $\phi(a, b) = \int_a^b \frac{\sin t}{t}\,\mathrm{d}t$ is continuous and bounded. [*Hint: We remind the following result that can be used without proof: the improper integral* $\int_0^{+\infty} \frac{\sin t}{t}\,\mathrm{d}t$ *exists and is* $\pi/2$.]
4. Show that the Fourier transform $\mathcal{F}g_k$ of $g_k := \chi_{1/k<|x|<k}\frac{1}{\pi x}$ for $k \geqslant 1$, is well-defined and bounded independently of k, and converges pointwise to a certain function g.
5. Prove that if $f \in L^2(\mathbb{R})$, the convolution $f * g_k$ converges in $L^2(\mathbb{R})$ to a function $H(f) \in L^2(\mathbb{R})$ (called the *Hilbert transform* of f).
6. Prove that $\|H(f)\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$ and $H(H(f)) = -f$.

(*) *Bonus:* ***not*** *needed to get full mark on the question.* Consider $f \in L^1(\mathbb{R})$ so that $F_y(x) := N(x, y)f(x)$ is integrable for almost every $y \in \mathbb{R}$. Prove that f is zero almost everywhere. [*Hint: Use Lesbegue's differentiation Theorem.*]

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Recalls: A topological space is *separable* if it contains a countable dense subset. The *dual* E' of a normed vector space E is the space of continuous linear forms on E.

1. Give (*without proof)* a countable dense subset of $L^p(\mathbb{R})$ when $p \in [1+\infty)$.
2. Prove that $L^\infty(\mathbb{R})$ is *not* separable.
3. Prove that if the dual E' of normed vector space E is separable then E itself is separable. [*Hint: Use the Hahn-Banach Theorem.*]
4. Prove that $L^1(\mathbb{R})$ is not the dual space of $L^\infty(\mathbb{R})$.
5. Recall what is the *generalised derivative* $D(f)$ of a function $f \in L^2(\mathbb{R})$.
6. For $f \in L^2(\mathbb{R})$ and $h > 0$ define $\tau_h f \in L^2(\mathbb{R})$ by $\tau_h f(x) := f(x+h)$. Assume that there is $C > 0$ s.t. $\|\tau_h f - f\|_{L^2(\mathbb{R})} \leqslant C|h|$ for all $h > 0$, then prove that $D(f)$ is an $L^2(\mathbb{R})$ function and $\frac{\tau_h f - f}{h}$ converges to $D(f)$ in the weak $L^2(\mathbb{R})$ topology as $h \to 0$.