**ANALYSIS OF FUNCTIONS (PART II)**
**EXAMPLE SHEET 3**

Harder questions are hightlighted with * and facultative "cultural" questions with %.
Focus first on questions 1 to 13 in priority for the supervision.

**Exercise 1.** Consider $f \in C_c^\infty(\mathbb{R})$ and prove that $\mathcal{F}(f)$ is $C^\infty$ and is the sum on $\mathbb{R}$ of an entire series with infinite radius of convergence.

**Exercise 2.** Consider $f \in \mathcal{S}(\mathbb{R}^d)$ i.e. $f \in C^\infty$ and $\forall\, \alpha \in \mathbb{N}^d,\ \beta \in \mathbb{N},\ |x|^\beta \partial^\alpha f(x) \to 0$ as $|x| \to +\infty$. In particular $f \in L^1(\mathbb{R}^d)$ and prove in full details that $\mathcal{F}(f) \in \mathcal{S}(\mathbb{R}^d)$.

**Exercise 3.** Recall that $\mathcal{F}$ has been extended in lecture to $L^2(\mathbb{R}^d)$ by establishing a bound from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$[1]. It is natural to ask whether one can extend by continuity the Fourier transform $\mathcal{F}$ to $L^p(\mathbb{R}^d)$ with $p \in (2, +\infty]$. This exercise answers negatively.
(i) Prove that if $p \in (2, +\infty)$ and $q \in [1, +\infty]$ are s.t. there is $C > 0$ s.t. for all $f \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ it holds $\|\mathcal{F}(f)\|_{L^q(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}$, then necessarily $q = p' = p/(p-1)$ is conjugate to $p$.
(ii) By computing the Fourier transform of the complex-valued function $f(x) = e^{-(a+ib)|x|^2}$ for $a > 0$ and $b \in \mathbb{R}$, prove that such inequality cannot hold when $p \in (2, +\infty]$.

***Exercise 4.** Consider $f \in L^1(\mathbb{R})$ that is differentiable almost everywhere with $f' \in L^1(\mathbb{R})$, does it imply that $\mathcal{F}(f')(\xi) = 2i\pi\xi\mathcal{F}(f)(\xi)$? [Prove it if true or give a detailed counter-example if not.]

**Exercise 5.** *Non-surjectivity of* $\mathcal{F} : L^1(\mathbb{R}^d) \to C_0(\mathbb{R}^d)$.
(i) Let $0 < a < b < +\infty$, and $f \in L^1(\mathbb{R}^d)$ odd ($f(-x) = -f(x)$). Prove that

$$\int_a^b \frac{\mathcal{F}f(\xi)}{\xi}\, \mathrm{d}\xi = -2i \int_0^{+\infty} f(x) \left( \int_{ax}^{bx} \frac{\sin(2\pi u)}{u}\, \mathrm{d}u \right) \mathrm{d}x.$$

(ii) Deduce that the improper integral $\int_0^{+\infty} \frac{\mathcal{F}f(\xi)}{\xi}\, \mathrm{d}\xi$ converges, and compute it.
(iii) Prove that if $g \in C_0(\mathbb{R})$ is the Fourier transform of $f \in L^1(\mathbb{R})$ and $g$ odd then $f$ odd.
(iv) By considering $g(\xi) := (1 + |\ln \xi|)^{-1}$ for $\xi > 0$, deduce that the Fourier transform from $L^1(\mathbb{R})$ to $C_0(\mathbb{R})$ is not surjective.

**Exercise 6.** Consider $f \in L^p(\mathbb{R}^d)$ and $g \in L^{p'}(\mathbb{R}^d)$ with $p \in [1, +\infty]$ and $p' = p/(p-1) \in [1, +\infty]$, and prove that $h := f * g$ is bounded and uniformly continuous. When $p \in (1, +\infty)$ prove moreover that $h(x) \to 0$ as $|x| \to +\infty$ and show that the latter fails when $p = 1$ or $p = +\infty$.

***Exercise 7.** State and prove a version of the Poisson summation formula suitable to deduce as an application the identity $\frac{e^{2\pi\alpha}+1}{e^{2\pi\alpha}-1} = \frac{1}{\pi} \sum_{n\in\mathbb{Z}} \frac{\alpha}{\alpha^2+n^2}$ for $\alpha > 0$.

**Exercise 8.** Given $k \in \mathbb{N}$, prove that (1) for every $p \in [1, +\infty]$, the space $W^{k,p}(\mathbb{R}^d)$ is a Banach space, (2) for every $p \in (1, +\infty)$ it is reflexive, (4) for every $p \in [1, +\infty)$ it is separable, (4) for $p = 2$ it is a Hilbert space.

---

*Date*: Lent 2017 – Feedbacks and comments welcome: `C.Mouhot@dpmms.cam.ac.uk`.

[1] As we will see in facultative questions the bounds $L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ yield more generally bounds $L^p(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d)$ for $p \in (1, 2)$ by interpolation.

**Exercise 9.** Prove that if $f \in L^1(\mathbb{R}^d)$ is invariant under rotations (i.e. depends only on the Euclidean distance to the origin) then the same is true for $\mathcal{F}(f)$. Given $f \in C_c^\infty(\mathbb{R}^d)$ invariant under rotation prove by using the Fourier transform that $\Delta f$ is also invariant under rotation; extend this result to $f \in H^2(\mathbb{R}^d)$ when $\Delta f$ is the generalised derivative Laplacian of $f$.

**Exercise 10.** $B(0,1)$ denotes the open unit ball of $\mathbb{R}^d$.
(i) Consider an open set $U \subset \mathbb{R}^d$ connected and $f \in W^{1,p}(U)$ so that all first-order generalised derivatives $D_{x_i} f = 0$ almost everywhere on $U$. Prove that $f$ is constant almost everywhere on $U$.
(ii) Consider an open set $U \subset \mathbb{R}^d$ and $f \in L^p(U)$ s.t $D^\alpha f \in L^p(U)$ for $|\alpha| = k \geq 2$, then is it true that $D^\alpha f \in L^p(U)$ for all $|\alpha| = 1, 2, \dots, k-1$?
(iii) Consider $F : \mathbb{R} \to \mathbb{R}$ that is $C^1$ and with $F'$ bounded, and $f \in W^{1,p}(B(0,1))$ for $p \in [1, +\infty]$ then prove that $F(f) \in W^{1,p}(B(0,1))$.
(iv) Prove that $f(x) := \ln \ln(1 + |x|^{-1}) \in W^{1,d}(B(0,1))$.
(v) For $s \in (0, 1/2)$ exhibit a $f \in H^s(\mathbb{R})$ that is not continuous (i.e. has no continuous representant).
(vi) Exhibit a function $f \in H^1(\mathbb{R}^2)$ that is not bounded.
*(vii) Exhibit an open set $U \subset \mathbb{R}^d$ and $f \in W^{1,\infty}(U)$ s.t. $f$ is not Lipschitz continuous on $U$.

**Exercise 11.** Prove that $f \in L^2(\mathbb{R}^d)$ belongs to $H^1(\mathbb{R}^d)$ iff there is a $\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^{\frac{s}{2}} \, \mathrm{d}\xi < +\infty$ where $\hat{f}$ is the Fourier-Plancherel transform of $f$, and the square root of this integral defines an equivalent norm on $H^s(\mathbb{R}^d)$. Deduce a proof based on the Fourier transform of the following Sobolev inequality: when $s > d/2$ there is $C > 0$ depending only on $d$ and $s$ s.t. $\|f\|_{L^\infty(\mathbb{R}^d)} \leq C \|f\|_{H^s(\mathbb{R}^d)}$.

***Exercise 12.** Consider $\alpha > 0$ and $f \in H^1(B(0,1))$ with $B(0,1)$ open unit ball of $\mathbb{R}^d$ s.t. $\lambda(\{x \in B(0,1) \mid f(x) = 0\}) \geq \alpha$ ($\lambda$ Lebesgue measure), then prove that there is $C > 0$ depending only on $d$ and $\alpha$ s.t. $\int_{B(0,1)} f^2 \, \mathrm{d}x \leq C \int_{B(0,1)} |\nabla f|^2 \, \mathrm{d}x$, where $\nabla f$ is the generalised derivative gradient.

**Exercise 13.** Denote by $U := B(0,1)$ the open unit ball of $\mathbb{R}^d$. Prove that $\|\Delta f\|_{L^2(U)}$ (where $\Delta f$ is the generalised derivative Laplacian) defines a norm on $H_0^2(U)$, equivalent to the ambient norm.

**Exercise 14.** Consider $p \in [1, +\infty]$ and $U \subset \mathbb{R}^d$ a bounded open set and $f \in W^{1,p}(U)$.
(i) Prove that $|f| \in W^{1,p}(U)$.
(ii) Prove that $f^+ := \max(f, 0)$ and $f^- := \max(-f, 0)$ belong to $W^{1,p}(U)$.
(iii) Calculate the generalised derivatives of $f^+$ and $f^-$ in terms of those of $f$ and deduce that $\nabla f = 0$ almost everywhere on the set $\{f = 0\}$, where $\nabla f$ is the gradient of generalised derivatives.

***Exercise 15.** *Dirichlet Principle*
Let $U \in \mathbb{R}^d$ be open and bounded. For a source term $g \in L^2(U)$, show that solving for $f \in H_0^1(U)$ the Dirichlet problem $-\Delta f = g$ in $U$ and $f = 0$ on $\partial U$ is the same as solving for $f \in H_0^1(U)$ the minimization problem: $F(f) = \inf_{h \in H_0^1(U)} F(h)$ where $F(h) = \frac{1}{2} \int_U |\nabla h|^2 \, \mathrm{d}x - \int_U hg \, \mathrm{d}x$.

***Exercise 16.** *Rellich-Kondrachov's Theorem*
Let $U \in \mathbb{R}^d$ open and bounded whose boundary $\partial U$ is $C^1$, prove that any sequence uniformly bounded in $H^1(U)$ is relatively compact in $L^2(U)$ i.e. if $\{f_n\} \subset H^1(U)$ is a sequence such that $\|f_n\|_{H^1(U)} \leq C$ for some constant $C$ independant of $n$, then there exists a subsequence $\{f_{\varphi(n)}\}$ (with $\varphi : \mathbb{N} \to \mathbb{N}$ strictly increasing) and a limit function $f \in L^2(U)$ such that $\|f_{\varphi(n)} - f\|_{L^2(\Omega)} \to 0$ as $n \to \infty$.