ANALYSIS OF FUNCTIONS (PART II)
EXAMPLE SHEET 2

Harder questions are hightlighted with * and facultative "cultural" questions with %.
Focus first on questions 1 to 11 in priority for the supervision.

Exercise 1. Let E Banach space. Consider F_n sequence of E' s.t. for any $f \in E$, the real sequence $F_n(f)$ converges, and prove that F_n converges weakly-* to some $F \in E'$ ($\sigma(E', E)$). Assume furthermore E reflexive and consider f_n sequence of E s.t. for any $F \in E'$, the real sequence $F(f_n)$ converges, and prove that f_n converges weakly to some $f \in E$ ($\sigma(E, E')$). Give an example of a non-reflexive Banach space where the latter does not hold.

Exercise 2. Let E Banach space.
(i) Consider $A \subset E$ a subset that is weakly-compact (i.e. for $\sigma(E, E')$). Prove A is bounded.
(ii) Consider $A \subset E$ convex, prove that its closure in the weak and strong topologies are the same.
(iii) Let E Banach space and f_n a sequence in E that converges weakly ($\sigma(E, E')$) to f, prove that $g_n := (f_1 + \cdots + f_n)/n$ converges weakly to f.
(iv) Prove that if f_n converges weakly to f and $\{f_n,\ n \geq 1\}$ is relatively compact for the strong topology, then f_n converges to f strongly.

Exercise 3. Let E Banach space, M subspace of E, $M^\perp := \{F \in E' \mid F(f) = 0\ \forall f \in M\}$, and $F_0 \in E'$. Prove that there is $G_0 \in M^\perp$ s.t. $\inf_{G \in M^\perp} \|F_0 - G\|_{E'} = \|F_0 - G_0\|$.

***Exercise 4.** Let E Banach space and f_n sequence of E. Define K_n the closure of the convex hull of $\{f_n, f_{n+1}, \dots\} = \cup_{i \geq n}\{f_i\}$. Prove that if f_n converges weakly to f ($\sigma(E, E')$) then $\cap_{n \geq 1} K_n = \{f\}$. Prove that if E is reflexive and the sequence f_n is bounded the converse holds: if $\cap_{n \geq 1} K_n = \{f\}$ then f_n converges weakly to f.

Exercise 5. Exhibit a sequence $f_n \in L^p(\mathbb{R})$, $p \in [1, +\infty)$ s.t. $\|f_n\|_{L^p} = 1$ for all $n \geq 1$ and f_n converges weakly to zero. Prove more generally that if [E Banach space reflexive] and/or [E Banach with E' separable], then there exists such a sequence.

Exercise 6. Prove that $f_n(x) = \sin(nx) \in L^2([0,1])$ converges weakly (give its limit) but not strongly in $L^2([0,1])$. Prove that $f_n(x) = \chi_{[n,n+1]}$ converges weakly (gives its limit) but not strongly in $L^2(\mathbb{R})$. Find a sequence f_n in $L^2(\mathbb{R}) \cap L^{3/2}(\mathbb{R})$ that converges to 0 in $L^2(\mathbb{R})$ weakly, to 0 in $L^{3/2}(\mathbb{R})$ strongly, but does not converge to 0 strongly in $L^2(\mathbb{R})$.

Exercise 7. Find a sequence of bounded measurable sets in $\mathbb{R}$ whose characteristic functions converge weakly in $L^2(\mathbb{R})$ to a function f with the property that $2f$ is a characteristic function. How about the possibility that $f/2$ is a characteristic function?

Exercise 8. Consider f_n a sequence bounded in $L^p(I)$ with $p \in (1, +\infty]$ and I bounded open interval, and s.t. $f_n \to f$ almost everywhere. Prove that $f_n \to f$ strongly in $L^q(I)$ for any $q \in [1, p)$.

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Exercise 9. *Relations between p-norms.*
(i) Given $1 \leq p \leq q \leq +\infty$ and $\Omega \subset \mathbb{R}$ open bounded, prove that $L^q(\Omega) \subset L^p(\Omega)$ with $\|f\|_{L^q(\Omega)} \geq C\|f\|_{L^p(\Omega)}$ for some constant $C > 0$.
(ii) Given $1 \leq p \leq q \leq +\infty$, prove that $\ell^p(\mathbb{R}) \subset \ell^q(\mathbb{R})$ with $\|f\|_{\ell^p(\mathbb{R})} \geq C\|f\|_{\ell^q(\mathbb{R})}$ for some $C > 0$.
(iii) Given $1 \leq p \leq r \leq q \leq +\infty$, prove that $L^p(\mathbb{R}) \cap L^q(\mathbb{R}) \subset L^r(\mathbb{R})$ and $\ell^p(\mathbb{R}) \cap \ell^q(\mathbb{R}) \subset \ell^r(\mathbb{R})$.
*(iv) Given $p, s \in [1, +\infty)$, prove that $L^p(\mathbb{R}) \cap \overline{B}_{L^s(\mathbb{R})}(0,1)$ is closed in $L^p(\mathbb{R})$ and prove that if $f_n \in L^p(\mathbb{R}) \cap \overline{B}_{L^s(\mathbb{R})}(0,1)$ converges strongly to f in $L^p(\mathbb{R})$ then it converges strongly to f in $L^r(\mathbb{R})$ for r between p an s, $r \neq s$.

***Exercise 10.** Given $p \in (1, +\infty)$ prove that there is $C > 0$ s.t. $|a-b|^p \leq C(|a|^p + |b|^p)^{1-p/2}(|a|^p + |b|^p - 2|(a+b)/2|^p)^{p/2}$ for all $a, b \in \mathbb{R}$. Deduce that $L^p(\mathbb{R})$ is uniformly convex for $p \in (1, 2]$.

Exercise 11. *Uniform convex spaces.*
Let E Banach space and $\mathcal{D} : E \to E'$ the duality (multi-valued) application $\mathcal{D}(f) = \{F \in E' \mid \|F\|_{E'} = \|f\|_E \text{ and } |F(f)| = \|f\|_E^2\}$. Assume E uniformly convex.
(i) Prove that for any $F \in E'$ there is a unique $f \in E$ s.t. $F \in \mathcal{D}(f)$ (inverse single-valued).
(ii) Prove that for any $\varepsilon > 0$ and $\alpha \in (0, 1/2)$ there is $\delta > 0$ s.t. for all $f, g \in \overline{B}_E(0,1)$ with $\|x-y\|_E \geq \varepsilon$ and $t \in [\alpha, 1-\alpha]$ then $\|tf + (1-t)g\|_E \leq 1 - \delta$.
*(iii) Prove that for C convex closed not empty the projection application $P_C(f)$ that realises $\inf_{g \in C} \|f - g\|_E$ is well-defined and uniformly continuous on bounded sets of E.

***Exercise 12.** Let E Banach space and $A \subset E$ closed in $\sigma(E, E')$, $B \subset E$ compact in $\sigma(E, E')$. Prove that $A + B$ is closed in $\sigma(E, E')$. Assume furthermore that A and B are not empty, convex and disjoint, then they can be separated stricly by a closed hyperplan.

***Exercise 13.** Construct a function in $\cap_{1 \leq p < +\infty} L^p((0,1))$ that is not in $L^\infty((0,1))$.

***Exercise 14.** Let $\gamma : \mathbb{R} \to \mathbb{R}$ measurable s.t. $\gamma f \in L^p(\mathbb{R})$ whenever $f \in L^q(\mathbb{R})$, with $1 \leq p \leq q \leq +\infty$. Prove that $\gamma \in L^r(\mathbb{R})$ with $r = pq/(q-p)$ if $q < +\infty$, and $r = p$ if $q = +\infty$.

***Exercise 15.** Consider a closed subspace S of $L^1(\mathbb{R})$ s.t.$S \subset \cup_{1 < q \leq +\infty} L^q(\mathbb{R})$. Prove that there is $p \in (1, +\infty]$ s.t. $S \subset L^p(\mathbb{R})$ and a constant $C > 0$ s.t. $\|f\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^1(\mathbb{R})}$ for all $f \in S$.

***Exercise 16.** Let $p \in [1, +\infty)$ and E be a closed subspace of $L^p([0,1])$ (for the strong topology) s.t. $E \subset L^\infty([0,1])$. Prove that E has finite dimension.

***Exercise 17.** Prove that in $\ell^1(\mathbb{R})$ a sequence converges strongly iff it converges weakly (i.e. in $\sigma(\ell^1, \ell^\infty)$. Is this statement true in $L^1(\mathbb{R})$?

***Exercise 18.** Consider f_n a sequence in $L^p(\mathbb{R})$ that converges weakly to $f \in L^p(\mathbb{R})$. Prove that there is a sequence $g_n := \sum_{i=1}^n c_i^n f_i$ for some $c_i^n \geq 0$ with $\sum_{i=1}^n c_i^n = 1$ (convex combination) s.t. g_n converges strongly to f in $L^p(\mathbb{R})$. When $p = 2$ prove furthermore that after passing to a subsequence the c_i^n's can be taken to be $c_i^n = 1/n$.

***Exercise 19.** Prove that $C^0([0,1])$ the space of continuous functions on $[0,1]$ endowed with the supremum norm is a Banach space that is not reflexive.

%Exercise 20. *Strictly convex spaces.*
A Banach space E is said *strictly convex* is its unit ball is a strictly convex set.
(i) Give an example of a Banach space that is not strictly convex.
(ii) Prove that if E is uniformly convex, it is strictly convex.
(iii) Prove that the converse is wrong: consider $\ell^1(\mathbb{R})$ with the modified norm $\|f\|_* = \|f\|_{\ell^1} + \|f\|_{\ell^2}$, prove that this norm is equivalent to the ℓ^1 norm, is strictly convex but not uniformly convex.
*(iv) Assume E is separable and build an equivalent norm on E that is strictly convex and s.t. the corresponding dual norm is also strictly convex in E'. [*Note that if E not reflexive, these norms cannot be uniformly convex.*]

%Exercise 21. *The Dunford-Pettis Theorem.*
(i) A function $g : [0,1] \to \mathbb{R}$ has *bounded variation* (BV) if the supremum over any subdivision $0 = x_0 \le x_1 \le \cdots \le x_n = 1$ of $\sum_{i=1}^n |g(x_i) - g(x_{i-1})|$ is finite. Prove that a BV function is the sum of two monotonous functions and that it is differentiable almost everywhere with f' integrable on $[0,1]$. Prove that a Lischitz function is BV. Give an example of a function that is *not* BV.
(ii) A function $g : [0,1] \to \mathbb{R}$ is *absolutely continuous* (AC) if for any $\varepsilon > 0$ there is $\delta > 0$ so that for any finite collection of pairwise disjoint subintervals (x_i, y_i), $i = 1, \dots, n$, with $\sum_{i=1}^n (y_i - x_i) \le \delta$, then $\sum_{i=1}^n |g(y_i) - g(x_i)| \le \varepsilon$. Prove that an AC function is BV and uniformly continuous. Prove that if f is AC, it is differentiable almost everywhere with $g' \in L^1([0,1])$, and *moreover* $g(y) - g(x) = \int_x^y g'$ for all $x, y \in [0,1]$.
(ii) Consider a sequence $f_n : [0,1] \to \mathbb{R}$ bounded in $L^1(\mathbb{R})$ and *uniformly integrable*: for any $\varepsilon > 0$ there is $\delta > 0$ s.t. for any measurable set $A \subset [0,1]$ with $\mu(A) < \delta$, $\int_A |f_n| < \varepsilon$ for all n. Define $F_n(x) := \int_0^x f_n$. Prove that the sequence F_n is equicontinuous and equicontinuous on $[0,1]$, and has a subsequence $F_{\theta(n)}$ that converges uniformly to some F and prove that this limit F is AC.
(iii) Prove that $\int_0^1 f_{\theta(n)} \chi_I \xrightarrow{n\to\infty} \int_0^1 f \chi_I$ for any interval $I \subset [0,1]$, where $f := F' \in L^1([0,1])$.
(iv) Deduce that $\int_0^1 f_{\theta(n)} \chi_A \xrightarrow{n\to\infty} \int_0^1 f \chi_A$ for any Borel set $A \subset [0,1]$.
(v) Deduce that $\int_0^1 f_{\theta(n)} s \xrightarrow{n\to\infty} \int_0^1 f s$ for any simple function s in $L^\infty([0,1])$.
(vi) Deduce that $f_{\theta(n)}$ converges to f in $\sigma(L^1, L^\infty)$.
(v) Extend this proof of the Dunford-Pettis Theorem to functions on $\mathbb{R}$ by assuming furthermore the *tightness* of the sequence: for any $\varepsilon > 0$ there is $M > 0$ s.t. $\int_{\mathbb{R}\setminus[-M,M]} |f_n| < \varepsilon$ for all n.

%Exercise 22. *No isomorphy between ℓ^p spaces.*
(i) Given $1 \le p < q < +\infty$ and $T : \ell^q(\mathbb{R}) \to \ell^p(\mathbb{R})$ linear continuous, prove that for any sequence f_n bounded in $\ell^p(\mathbb{R})$, the sequence $T(f_n)$ has a subsequence that converges strongly in $\ell^q(\mathbb{R})$ (Pitt's theorem of "automatic compactness").
(ii) Is this statement true in $L^p(\mathbb{R})$ / $L^q(\mathbb{R})$ spaces?
(iii) Deduce from (i) that there is no linear map bijective continuous and with continuous inverse between $\ell^p(\mathbb{R})$ and $\ell^q(\mathbb{R})$ for $1 \le p < q \le \infty$.