ANALYSIS OF FUNCTIONS (PART II)
EXAMPLE SHEET 1

Harder questions hightlighted with * and facultative "cultural" questions highlighted with %.

Exercise 1. Let A, B Borel sets of $\mathbb{R}^n$ with finite non-zero measure. Show the convolution $\chi_A * \chi_B(x) := \int_{\mathbb{R}^n} \chi_A(y)\chi_B(x-y)\,\mathrm{d}y$ of their two characteristic functions is a continuous function and is not the zero function. Deduce that $A + B$ contains a ball.

Exercise 2. Let $(A_k)_{k\geq 1}$ Borel sets of $\mathbb{R}^n$ s.t. $\sum_{k\geq 1} \mu(A_k) < +\infty$ (where μ is the Lebesgue measure). Prove that the set of points belonging to infinitely many A_k's has zero measure.

Exercise 3. Consider a sequence of measurable functions $f_k : \mathbb{R}^n \to \mathbb{C}$. Prove that the set of points where the sequence converges is measurable.

Exercise 4. Let $f : \mathbb{R}^n \to [0, +\infty]$ be a measurable function s.t. $\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x = 0$. Prove that f is zero almost everywhere.

Exercise 5. Let $f \in L^1(\mathbb{R}^n)$. Assume that $\int_{\mathbb{R}^n} f(x)\varphi(x)\,\mathrm{d}x = 0$ for all smooth compactly supported functions $\varphi : \mathbb{R}^n \to \mathbb{R}$. Prove that $\|f\|_{L^1(\mathbb{R}^n)} = 0$ and f is zero almost everywhere.

Exercise 6. Let $f \in L^1(\mathbb{R}^n)$. Prove that for any $\varepsilon > 0$ there is $\delta > 0$ s.t. $\int_E |f|\,\mathrm{d}x < \varepsilon$ as soon as $\mu(E) < \delta$ (where μ denotes the Lebesgue measure).

Exercise 7. Give a counter-example showing that the domination assumption is necessary in Lebesgue's dominated convergence Theorem.

***Exercise 8.** Consider $f_n : \mathbb{R}^n \to \mathbb{R}_+$ a sequence of measurable functions converging pointwise to f on $\mathbb{R}^n$, and s.t. $\int_{\mathbb{R}^n} f_n\,\mathrm{d}x \leq \int_{\mathbb{R}^n} f\,\mathrm{d}x < +\infty$. Prove that $\lim_{n\to\infty} \int_{\mathbb{R}^n} |f_n - f|\,\mathrm{d}x = 0$.

Exercise 9. Let $f : \mathbb{R} \to \mathbb{R}_+$ measurable s.t. $\int_{\mathbb{R}} f\,\mathrm{d}x < +\infty$. Calculate $\lim_{n\to\infty} \int_{\mathbb{R}} n \ln(1 + (\frac{f}{n})^\alpha)\,\mathrm{d}x$ for $\alpha > 0$.

***Exercise 10.** Construct a sequence of continuous functions $f_n : [0, 1] \to \mathbb{R}_+$ that converges at no point, and whose integral converges to zero.

Exercise 11. Consider $f : \mathbb{R} \to [0, +\infty]$ measurable. Recall: Changing variables inside integrals can be used from *Probability & Measure* without proof.

(1) Assume $\int_{\mathbb{R}} f\,\mathrm{d}x < +\infty$, and define $g(x) := \sum_{k=-\infty}^{k=+\infty} f(x + k) \in [0, +\infty]$. Prove that $g(x)$ is finite for almost every $x \in \mathbb{R}$.
(2) Assume f is periodic and has finite integral on any compact set. Prove that $\lim_{n\to\infty} \frac{1}{n^2} f(nx) = 0$ for almost every $x \in \mathbb{R}$.

Date: Lent 2017 – Feedbacks and comments welcome: `C.Mouhot@dpmms.cam.ac.uk`.

Exercise 12. *Approximation of the unit and convolution.*

An *approximation of the unit* is a sequence of measurable functions $\varphi_k : \mathbb{R}^n \to \mathbb{R}_+$, $k \geq 1$, with $\int_{\mathbb{R}^n} \varphi_k(x)\,\mathrm{d}x = 1$ and support of φ_k included in $B(0, \varepsilon_k)$ for all $k \geq 1$, with $0 < \varepsilon_k \to 0$ as $k \to \infty$.

(1) Construct an example of such sequence. Can you assume furthermore that the φ_k are smooth?
(2) Prove that the *translation operator* is continuous in $L^1(\mathbb{R}^n)$: for $f \in L^1(\mathbb{R}^n)$ and $\tau_h f$ defined by $\tau_h f(x) = f(x+h)$, the convergence $\|\tau_h f - f\|_{L^1(\mathbb{R}^n)} \to 0$ as $h \to 0$ holds. [*Hint: Argue by density of simple functions in* $L^1(\mathbb{R}^n)$*.*]
(3) Deduce that if $f \in L^1(\mathbb{R}^n)$, the sequence $f * \varphi_k$ is well-defined, belongs to $L^1(\mathbb{R}^n)$ and converges to f in $L^1(\mathbb{R}^n)$.
(4) Give a new proof of the density of $C_c^\infty(\mathbb{R}^n)$ in $L^1(\mathbb{R}^n)$. [*Hint: Truncate* f *and convolute with a smooth approximation of the unit.*]

Exercise 13. Let $I = (0,1)$, $p \in [1, +\infty)$, $f \in L^p(\mathbb{R})$ where $f = 0$ outside I. Define $f_h(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(y)\,\mathrm{d}y$ for $h > 0$.

(1) Prove that f_h is well-defined for all $h > 0$.
(2) Prove that f_h is continuous.
(3) Prove that $\|f_h\|_{L^p(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})}$.
(4) Prove that $\|f_h - f\|_{L^p(\mathbb{R})} \to 0$ as $h \to 0$.

Exercise 14. Let E closed set of $\mathbb{R}^n$. The aim is to construct a non-negative smooth function on $\mathbb{R}^n$ s.t. $f(x) = 0$ iff $x \in E$.

(1) Construct $\varphi : \mathbb{R}^n \to \mathbb{R}_+$ smooth, positive on $B(0,1)$, that is zero outside $B(0,1)$ and s.t. $\int_{\mathbb{R}^n} \varphi(x)\,\mathrm{d}x = 1$.
(2) Let $\varphi_k(x) = \frac{1}{k^k}\varphi(kx)$, $k \geq 1$. Denote $V_\varepsilon = \{x \mid \mathrm{dist}(x, E) < \varepsilon\}$, where $\mathrm{dist}(x, E) = \inf\{|x - y| \mid y \in E\}$. Let χ_k the characteristic function of $\mathbb{R}^n \setminus V_{2/k}$ and $f_k := \varphi_k * \chi_k$. Prove that f_k is smooth, $f_k \equiv 0$ on $V_{1/k}$, $f_k > 0$ on $\mathbb{R}^n \setminus V_{3/k}$.
(3) Calculate a supremum bound on all partial derivatives of f_k.
(4) Conclude by summing the f_k's.

***Exercise 15.** *Study of the convergence in measure.*

Take a sequence of measurable functions $f_n : [0,1] \to \mathbb{C}$. The sequence *converges in measure* to f if for any $\varepsilon > 0$, $\mu(\{x \in \mathbb{R} \mid |f(x) - f_n(x)| > \varepsilon\})$ goes to zero as $n \to \infty$.

(1) Prove that if f_n converges almost everywhere to f on $[0,1]$, it converges in measure to f on $[0,1]$. Is the reciprocal statement true?
(2) Prove that if f_n converges to f in $L^p([0,1])$, $p \in [1, +\infty]$, then it converges in measure to f on $[0,1]$. Is the reciprocal statement true?
(3) Prove that if f_n converges in measure to f, then there is a subsequence of f_n converging almost everywhere to f.
(4) Are the results (1) and (2) true on $\mathbb{R}$ instead of $[0,1]$?

***Exercise 16.** *Study of the ergodic average.*

Let $p \in (1, +\infty)$ and $f \in L^p((0, +\infty))$, and denote $T[f](x) = \frac{1}{x} \int_0^x f(y)\,\mathrm{d}y$ for $x \in (0, +\infty)$.

(1) Prove that the function $T[f]$ is well-defined on $(0, +\infty)$.
(2) Assume f continuous with compact support in $(0, +\infty)$.
    (a) Give a differential equation expressing f in terms of $T[f]$.
    (b) Prove $\|T[f]\|_{L^p((0,+\infty))} \leq \frac{p}{p-1} \|f\|_{L^p((0,+\infty))}$. [*Hint: Assume first that* f *is non-negative, and use integration by parts and (b). Relax then the assumption of non-negativity by linearity.*]
(3) Prove that if $f_n \to f$ in $L^p((0, +\infty))$, then $T[f_n](x)$ converges to $T[f](x)$ for all $x \in (0, +\infty)$.

(4) Prove $\|T[f]\|_{L^p((0,+\infty))} \leq \frac{p}{p-1}\|f\|_{L^p((0,+\infty))}$ for all $f \in \mathrm{L}^p((0,+\infty))$ and compute the norm of the linear map $T : L^p((0,+\infty)) \to L^p((0,+\infty))$.
(5) Does T map $L^1((0,+\infty))$ into $L^1((0,+\infty))$?

%**Exercise 17.** *A theorem due to Lebesgue on Riemann-integrable functions.*

Recall from Riemann's integrability theory: We consider $f : [a,b] \to \mathbb{R}$ bounded. Let $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a partition of $[a,b]$ and define $\ell_{\mathcal{P}} := \sum_{i=1}^n m_i \chi_{(t_i-1,t_i]}$ and $u_{\mathcal{P}} := \sum_{i=1}^n M_i \chi_{(t_i-1,t_i]}$, where $m_i := \inf\{f(x) \ : \ x \in [t_i-1,t_i]\}$ and $M_i := \sup\{f(x) \ : \ x \in [t_i-1,t_i]\}$. Denote $\int_a^b \ell_{\mathcal{P}} = L(f,\mathcal{P})$ and $\int_a^b u_{\mathcal{P}} = U(f,\mathcal{P})$. Riemann-integrability of f holds when lower and upper Darboux sums both converge to $L(f) = U(f)$ when the mesh size of the partition goes to zero. This limit does not depend on the choice of sequence of partitions, and is called the Riemann integral of f.

(1) Prove that if $f : [a,b] \to \mathbb{R}$ bounded is Riemann-integrable, it is measurable and belongs to $L^1([a,b])$.
(2) Define $H(x) := \inf_{\delta>0} \sup\{f(y) \ : \ y \in [a,b] \text{ and } |y-x| \leq \delta\}$ and $h(x) := \sup_{\delta>0} \inf\{f(y) \ : \ y \in [a,b] \text{ and } |y-x| \leq \delta\}$. Prove that $h(x) \leq f(x) \leq H(x)$ for all $x \in [a,b]$, and f is continuous at x iff $h(x) = H(x)$.
(3) Consider an increasing (i.e. adding more points at each step) sequence of partitions $\mathcal{P}_k$ with mesh size less than $1/k$, and denote $u = \inf_k u_{\mathcal{P}_k} = \lim_k u_{\mathcal{P}_k}$ and $\ell = \sup_k \ell_{\mathcal{P}_k} = \lim_k \ell_{\mathcal{P}_k}$. Show that these functions are well-defined (including that the limits do exist), satisfy $\ell(x) \leq f(x) \leq u(x)$ for all $x \in [a,b]$, and prove that f Riemann-integrable iff $u = \ell = f$ almost everywhere.
(4) Denote $N = \cup \mathcal{P}_k$ the set of all points defining the partitions (countable with measure zero). Show that $H(x) = u(x)$ and $h(x) = \ell(x)$ on $x \in [a,b] \setminus N$.
(5) Deduce that h and H are measurable and prove that f Riemann-integable iff it is continuous almost everywhere.
(6) Compare with Lusin's Theorem for measurable functions.

%**Exercise 18.** *Arzéla's dominated convergence Theorem for Riemann-integrable functions.*

(1) Is the statement of Lebesgue's dominated convergence Theorem correct for bounded functions $f_n : [0,1] \to \mathbb{R}$ when "L^1" is replaced by "Riemann-integrable"?
(2) Consider a nonincreasing sequence of continuous functions $p_n : [0,1] \to \mathbb{R}$ ($p_{n+1} \leq p_n$) that converges pointwise to zero, prove that the convergence is uniform.
(3) Consider $f_n : [0,1] \to \mathbb{R}$ Riemann-integrable with $|f_n| \leq 1$ and converging pointwise to zero, and define $g_n(x) := \sup_{m \geq n} |f_m(x)|$. Given $\varepsilon > 0$, construct h_n continuous s.t. $0 \leq h_n \leq g_n$ and $L(g_n) \leq \int_0^1 h_n + \frac{\varepsilon}{2^n}$ (the lower Darboux integral $L(g_n)$ is defined in the previous exercise).
(4) Define $p_n(x) = \min\{h_1(x), \ldots, h_n(x)\}$ and prove using (2) that $\int_0^1 p_n$ goes to zero as $n \to \infty$.
(5) Prove that $h_n \leq p_n + \sum_{j=1}^{n-1}(g_j - h_j)$.
(6) Deduce that $\limsup_{n\to\infty} \int_0^1 h_n \leq \varepsilon$.
(7) Deduce that $\limsup_{n\to\infty} L(g_n) \leq \varepsilon$ and finally $\limsup_{n\to\infty} \int_0^1 |f| \leq \varepsilon$.
(8) Prove the following theorem: *Consider a sequence of bounded Riemann-integrable functions $f_n : [0,1] \to \mathbb{R}$ that converges pointwise to a bounded Riemann-integrable function f and satisfies $|f_n| \leq F$ with F bounded Riemann-integrable, then the Riemann integral $\int_0^1 f_n \to \int_0^1 f$ as $n \to \infty$.*