1 Analysis of Functions

(a) Consider a measure space \((X, \mathcal{A}, \mu)\) and a complex-valued measurable function \(F\) on \(X\). Prove that for any \(\varphi : [0, +\infty) \to [0, +\infty)\) differentiable and increasing such that \(\varphi(0) = 0\), then

\[
\int_X \varphi(|F(x)|) \, d\mu(x) = \int_0^{+\infty} \varphi'(s)\mu(|F| > s) \, d\lambda(s)
\]

where \(\lambda\) is the Lebesgue measure.

(b) Consider a complex-valued measurable function \(f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\) and its maximal function \(Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f| \, d\lambda\). Prove that for \(p \in (1, +\infty)\) there is a constant \(c_p > 0\) such that \(\|Mf\|_{L^p(\mathbb{R}^n)} \leq c_p \|f\|_{L^p(\mathbb{R}^n)}\). [Hint: Split \(f = f_0 + f_1\) with \(f_0 = f\chi_{|f|>s/2}\) and \(f_1 = f\chi_{|f|\leq s/2}\) and prove that \(\lambda(|\{Mf > s\}|) \leq \lambda(|\{Mf_0 > s/2\}|)\). Use then the maximal inequality \(\lambda(|\{Mf > s\}|) \leq \frac{C_1}{s} \|f\|_{L^1(\mathbb{R}^n)}\) for some constant \(C_1 > 0\).]

(c) Consider \(p, q \in (1, +\infty)\) with \(p < q\) and \(\alpha \in (0, n)\) such that \(1/q = 1/p - \alpha/n\). Define \(I_\alpha[f](x) := \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^\alpha} \, d\lambda(y)\) and prove \(I_\alpha[f](x) \leq \|f\|_{L^p(\mathbb{R}^n)}^{\alpha/p} Mf(x)^{1-\alpha/p}\). [Hint: Split the integral into \(|x-y| \geq r\) and \(|x-y| \in [2^{-k-1}r, 2^{-k}r)\) for all \(k \geq 0\), given some suitable \(r > 0\).]

[Note that the latter implies the Hardy-Littlewood-Sobolev inequality: given \(p, q \in (1, +\infty)\) with \(p < q\) and \(\alpha \in (0, n)\) such that \(1/q = 1/p - \alpha/n\) there is \(C_{HLS} > 0\) so that \(\|I_\alpha f\|_{L^q(\mathbb{R}^n)} \leq C_{HLS} \|f\|_{L^p(\mathbb{R}^n)}\) for any \(f \in L^p(\mathbb{R}^n)\).]

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1. Given \(X = \mathbb{R}^n\) endowed with the Borel sets and Lebesgue measure, define the \(L^p(X)\) spaces for \(p \in [1, +\infty]\). Describe the dual spaces of \(L^p(X)\) for \(p \in [1, +\infty)\). Define "reflexivity" and say which \(L^p(X)\) are reflexive. Prove that \(L^1(X)\) is not the dual space of \(L^\infty(X)\).

2. Consider now a general measure space \((X, \mathcal{A}, \mu)\). Prove that if \(\mu(X) < +\infty\) then \(L^q(X) \subset L^p(X)\) for all \(1 \leq p \leq q \leq +\infty\).

3. Assume that \(X \subset \mathbb{R}^n\) measurable endowed with Borel sets and Lebesgue measure.

   (a) Given any \(p \in [1, +\infty]\), prove that any sequence \((f_n)\) of \(L^p(X)\) converging in \(L^p(X)\) to some \(f \in L^p(X)\) admits a subsequence converging almost everywhere to \(f\).

   (b) Prove that if \(L^q(X) \subset L^p(X)\) for \(1 \leq p < q \leq +\infty\) then \(\mu(X) < +\infty\). [Hint: You might want to prove first that the inclusion is continuous with the help of one of the corollaries of Baire’s category theorem.]

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Here and below, \(\Phi : \mathbb{R} \to \mathbb{R}\) is smooth such that \(\int_{\mathbb{R}} e^{-\Phi(x)} \, dx = 1\) and

\[
\lim_{|x| \to +\infty} \left( \frac{|\Phi'(x)|^2}{4} - \frac{\Phi''(x)}{2} \right) = \ell \in (0, +\infty).
\]
$C^1_c(\mathbb{R})$ denotes the continuously differentiable functions with compact support on $\mathbb{R}$.

(a) Prove that there is an $R_0 > 0$, $\lambda_1 > 0$ and $K_1 > 0$ so that for any $R \geq R_0$ and $h \in C^1_c(\mathbb{R})$:

$$\int_{\mathbb{R}} \left| h'(x) \right|^2 e^{-\Phi(x)} \, dx \geq \lambda_1 \int_{\{x \geq R\}} |h(x)|^2 e^{-\Phi(x)} \, dx - K_1 \int_{\{x \leq R\}} |h(x)|^2 e^{-\Phi(x)} \, dx.$$  

[Hint: Denote $g := he^{-\Phi/2}$, expand the square and integrate by parts.]

(b) Prove that, given any $R > 0$, there is a $C_R > 0$ so that for any $h \in C^1([-R, R])$ with $\int_{-R}^R he^{-\Phi} = 0$:

$$\max_{x \in [-R, R]} |h(x)| + \sup_{\{x, y \in [-R, R], x \neq y\}} \frac{|h(x) - h(y)|}{|x - y|^{1/2}} \leq C_R \left( \int_{-R}^R |h'(x)|^2 e^{-\Phi} \, dx \right)^{1/2}.$$  

[Hint: Use the fundamental theorem of calculus to control the second term of the left-hand side, and then compare $h$ to its mean to control the first term of the left-hand side.]

(c) Prove that, given any $R > 0$, there is a $\lambda_R > 0$ so that for any $h \in C^1([-R, R])$:

$$\int_{-R}^R |h'(x)|^2 e^{-\Phi(x)} \, dx \geq \lambda_R \int_{-R}^R \left| h(x) - \int_{-R}^R h(y)e^{-\Phi(y)} \, dy \right|^2 e^{-\Phi(x)} \, dx.$$  

[Hint: Show first that one can reduce to the case $\int_{-R}^R he^{-\Phi} = 0$. Then argue by contradiction with the help of Arzelà–Ascoli theorem and part (b).]

(d) Deduce that there is a $\lambda_0 > 0$ so that for any $h \in C^1_c(\mathbb{R})$:

$$\int_{\mathbb{R}} |h'(x)|^2 e^{-\Phi(x)} \, dx \geq \lambda_0 \int_{\mathbb{R}} \left[ h(x) - \left( \int_{\mathbb{R}} h(y)e^{-\Phi(y)} \, dy \right) \right]^2 e^{-\Phi(x)} \, dx.$$  

[Hint: Show first that one can reduce to the case $\int_{\mathbb{R}} he^{-\Phi} = 0$. Then combine the inequality (a) multiplied by a small constant $\epsilon = \epsilon_0 \lambda_R$ with a small $\epsilon_0 > 0$, and the inequality (c).]