4. Generalised derivatives of functions

4.1. **Measuring regularity by integrals.** Our usual way to measure regularity so far is through:

Definition 4.1. *We define $C^\theta(\mathbb{R}^d)$ for $\theta \in \mathbb{R}_+$ as a subspace of $C^{[\theta]}$, where $[\theta]$ is the integer part of θ, where the $[\theta]$-order derivatives are $(\theta - [\theta])$ Hölder-continuous. It is endowed with the norm*

$$\|u\|_{C^\theta} := \sum_{|\alpha| \le [\theta]} \|\partial^\alpha u\|_{L^\infty(\mathbb{R}^d)} + \sum_{|\alpha| = [\theta]} \sup_{x \neq y \in \mathbb{R}^d} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^{\theta - [\theta]}}.$$

Exercise 47. *Prove that for any $\theta \in \mathbb{R}_+$ the space $C^\theta(\mathbb{R}^d)$ is a Banach space.*

The motivation of searching for other spaces is to measure regularity by means of *integrals* (as opposed to pointwise as for C^k spaces) which is natural for PDEs (energy, entropy in physics, etc.).

Definition 4.2 (First definition of Sobolev spaces)**.** *Given for $s \in \mathbb{N}$ and $p \in [1, +\infty)$, we define the **Sobolev space** $W^{s,p}(\mathbb{R}^d)$ on $\mathbb{R}^d$, as a subspace of $L^p(\mathbb{R}^d)$, by building the completion of $C_c^\infty(\mathbb{R}^d)$ endowed with the norm $\|g\|_{W^{s,p}(\mathbb{R}^d)} := (\sum_{|\alpha| \le s} \|\partial_x^\alpha g\|^2_{L^p(\mathbb{R}^d)})^{\frac{1}{2}}$ within $L^p(\mathbb{R}^d)$. It means $W^{s,p}(\mathbb{R}^d) = \overline{C_c^\infty(\mathbb{R}^d)}^{\|\cdot\|_{W^{s,p}(\mathbb{R}^d)}} \subset L^p(\mathbb{R}^d)$ is the closure of $C_c^\infty(\mathbb{R}^d)$ within $L^p(\mathbb{R}^d)$ for the norm $\|\cdot\|_{W^{s,p}(\mathbb{R}^d)}$. We write $H^s(\mathbb{R}^d) := W^{s,2}(\mathbb{R}^d)$ in the case $p = 2$.*

Exercise 48. *Given $p \in [1, +\infty)$ prove that $W^{s,p}(\mathbb{R}^d)$ is a Banach space, and $C_c^\infty(\mathbb{R}^d)$ is dense in it.*

Exercise 49. *In the case $p = 2$ we can give two other definitions: (1) $g \in L^2(\mathbb{R}^d)$ belongs to $H^s(\mathbb{R}^d)$ iff there is a constant $C > 0$ so that*

$$\forall \varphi \in C_c^\infty(\mathbb{R}^d),\ \forall |\alpha| \le s, \qquad \left| \int_{\mathbb{R}^d} g(x) \partial_x^\alpha \varphi(x) \, \mathrm{d}x \right| \le C \|\varphi\|_{L^2(\mathbb{R}^d)}$$

and the smaller such constant is precisely the $H^s(\mathbb{R}^d)$ norm of g. (2) $g \in L^2(\mathbb{R}^d)$ belongs to $H^s(\mathbb{R}^d)$ iff there is a constant $C > 0$ so that $\left(\int_{\mathbb{R}^d} |\hat{g}(\xi)|^2 (1 + |\xi|^2)^{\frac{s}{2}} \, \mathrm{d}\xi\right)^{\frac{1}{2}} \le C$ where $\hat{g}$ is the Fourier-Plancherel transform of g, and the smaller such constant is precisely the $H^s(\mathbb{R}^d)$ norm of g.

(Remark that this last definition allows to consider non-integer $s \in \mathbb{R}_+$, another way to define H^s for non-integer s would be to use the interpolation theory). Check that all the three previous definitions are equivalent for $H^s(\mathbb{R}^d)$ and provide a Hilbert space, which is dense in $L^2(\mathbb{R}^d)$.

Remark 4.3. *For $g \in W^{1,p}(\mathbb{R}^d)$ with $p \in [1, +\infty)$, this defines thus a **generalised (or weak) derivative** "$\nabla g \in L^p(\mathbb{R}^d)$" as the limit in $L^p(\mathbb{R}^d)$ of ∇g_n where $g_n \in C_c^\infty(\mathbb{R}^d)$ approximates g in $W^{1,p}(\mathbb{R}^d)$. In general this generalised derivative is **not** related to g by the standard differential calculus, that is in a pointwise sense. However (check it by limit) it satisfies the integration by parts as follows:*

$$\forall \varphi \in C_c^\infty(\mathbb{R}^d), \qquad \int_{\mathbb{R}^d} \nabla g \varphi \, \mathrm{d}x = -\int_{\mathbb{R}^d} g \nabla \varphi \, \mathrm{d}x.$$

This motivates the following more general definition:

Definition 4.4 (Generalised derivative)**.** *Consider f locally integrable (i.e. integrable on any compact set) on $\mathbb{R}^d$ and $\alpha \in \mathbb{N}^d$ multi-index. We say that g is the α-th generalised partial derivative of f, written $g := D^\alpha f$, if for all $\varphi \in C_c^\infty(\mathbb{R}^d)$ one has $\int_{\mathbb{R}^d} f D^\alpha \varphi \, \mathrm{d}x = (-1)^{|\alpha|} \int_{\mathbb{R}^d} g \varphi \, \mathrm{d}x$.*

Exercise 50. *The generalised derivative is unique when it exists. It satisfies Leibniz formula when it exists. What is the generalised derivative of $f(x) = |x|$ on $\mathbb{R}$?*

Definition 4.5 (Second definition of Sobolev spaces)**.** *Given $s \in \mathbb{N}$ and $p \in [1, +\infty]$, we define the **Sobolev space** $W^{s,p}(\mathbb{R}^d)$ on $\mathbb{R}^d$, as a subspace of functions of $L^p(\mathbb{R}^d)$ s.t. for any $\alpha \in \mathbb{N}^d$ with $|\alpha| \le s$ the generalised derivative $D^\alpha f$ exists and belongs to $L^p(\mathbb{R}^d)$. It is a Banach space when endowed with the norm described above when $p \in [1, +\infty)$ and the norm $\sum_{|\alpha| \le s} \|D^\alpha f\|_{L^\infty(\mathbb{R}^d)}$ when $p = +\infty$.*

Exercise 51. *Prove when $p \in [1, +\infty)$ that the first and second definition are equivalent.*

4.2. **Relating integral and pointwise regularity.** Sobolev embedding is the fundamental tool to relate both ways to measure regularity, and writes

$$\forall\, s \neq (d/2)\mathbb{N},\ s \in \mathbb{N},\ s > d/2,\ \exists\, C > 0, \quad \|u\|_{C^{s-d/2}(\mathbb{R}^d)} \le C\|u\|_{H^s(\mathbb{R}^d)}.$$

We consider here real-valued functions. We start with dimension $d = 1$.

Proposition 4.6. *We have $H^1(\mathbb{R}) \subset C^{1/2}(\mathbb{R})$ (i.e. any function of $H^1(\mathbb{R})$ has a representant for the almost everywhere equality equivalence that is in $C^{1/2}(\mathbb{R})$), and there is $C > 0$ so that any $u \in H^1(\mathbb{R})$ satisfies $\|u\|_{C^{1/2}(\mathbb{R})} \le C\|u\|_{H^1(\mathbb{R})}$. As a consequence: $\forall\, s \ge 1,\ \exists\, C > 0,\ \|u\|_{C^{s-1/2}(\mathbb{R})} \le C\|u\|_{H^s(\mathbb{R})}$.*

Proof of Proposition 4.6. Consider first $u \in C^\infty_c(\mathbb{R})$:

$$u(x)^2 = 2\int_{-\infty}^x u(y)u'(y)\,\mathrm{d}y \Longrightarrow u(x)^2 \lesssim \|u\|_{L^2(\mathbb{R})}\|u'\|_{L^2(\mathbb{R})}$$

$$\Longrightarrow \|u\|_{L^\infty(\mathbb{R})} \lesssim \left(\|u\|^2_{L^2(\mathbb{R})} + \|u'\|^2_{L^2(\mathbb{R})}\right)^{1/2} = \|u\|_{H^1(\mathbb{R})}.$$

Moreover we can estimate variations as

$$u(x) - u(y) = \int_x^y u'(z)\,\mathrm{d}z \Longrightarrow |u(x) - u(y)| \lesssim |x - y|^{1/2}\|u'\|_{L^2(\mathbb{R})} \Longrightarrow \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^{1/2}} \lesssim \|u\|_{H^1(\mathbb{R})}$$

which proves that $\|u\|_{C^{1/2}(\mathbb{R})} \le C\|u\|_{H^1(\mathbb{R})}$. We then argue by density: consider $u \in H^1(\mathbb{R})$ and $u_n \in C^\infty_c(\mathbb{R})$ s.t. $\|u_n - u\|_{H^1(\mathbb{R})} \to 0$ as $n \to +\infty$. Then we have $\|u_m - u_n\|_{C^{1/2}(\mathbb{R})} \le C\|u_m - u_n\|_{H^1(\mathbb{R})} \to 0$ as $m, n \to +\infty$, hence (u_n) is Cauchy in $C^{1/2}(\mathbb{R})$, its limit u is also the limit in H^1 and taking the limit in $\|u_n\|_{C^{1/2}(\mathbb{R})} \le C\|u_n\|_{H^1(\mathbb{R})}$ gives the desired inequality $\|u\|_{C^{1/2}(\mathbb{R})} \le C\|u\|_{H^1(\mathbb{R})}$. $\square$

In higher dimension a simple formulation (sufficient for the study of elliptic regularity) is:

Proposition 4.7. *Given $d \ge 1$ integer and $k, s \in \mathbb{N}$, $s > k + d/2$, we have $C^k(\mathbb{R}^d) \subset H^s(\mathbb{R}^d)$, and there is $C > 0$ so that any $u \in H^s(\mathbb{R}^d)$ satisfies $\|u\|_{C^k(\mathbb{R}^d)} \le C\|u\|_{H^s(\mathbb{R}^d)}$.*

The more general result is:

Theorem 4.8. *[Sobolev inequalities] Given $s \in \mathbb{N}$ and $p \in [1, +\infty)$, the Sobolev space $W^{s,p}(\mathbb{R}^d)$ embeds continuously as follows:*
(1) in $L^q(\mathbb{R}^d)$ with $q \in [p, p/(1 - ps/d)]$ if $s < d/p$,
(2) in $L^q(\mathbb{R}^d)$ with any $q \in [p, +\infty)$ if $s = d/p$,
(3) in $C^{s-d/p}(\mathbb{R}^d)$ if $s > d/p$ and $s - d/p$ not integer.

The case $d = 1$ is proved by the previous proposition. In higher dimension, the proof uses the so-called Sobolev-Gagliardo-Nirenberg inequality.

Proposition 4.9. *Assume $d > p$. We have $W^{1,p}(\mathbb{R}^d) \subset L^{p^*}(\mathbb{R}^d)$ with $p^* := pd/(d - p)$ and there is $C > 0$ so that*

$$\forall\, u \in W^{1,p}(\mathbb{R}^d), \quad \|u\|_{L^{p^*}(\mathbb{R}^d)} \le C\|u\|_{W^{1,p}(\mathbb{R}^d)}.$$

Remark 4.10. *Observe that it implies $W^{1,p}(\mathbb{R}) \subset L^q(\mathbb{R})$ for $q \in [p, p^*]$ by Hölder's inequality.*

Proof of Proposition 4.9. We prove an intermediate result, the Sobolev-Gagliardo-Nirenberg inequality, that will allow us to use the one-dimensional argument in an "average way" on all variables:

Lemma 4.11. *Let $n \ge 2$ and $f_1, \dots, f_n : \mathbb{R}^{n-1} \to \mathbb{R}$ belonging to $L^{n-1}(\mathbb{R}^{n-1})$. For any $1 \le i \le n$ we denote $\tilde{x}_i = (x_1, \dots, x_{i-1}, x_i, \dots, x_n)$ (removing the i-th component), and $f(x) := f_1(\tilde{x}_1) \cdots f_n(\tilde{x}_n)$. Then f is integrable with*

$$\|f\|_{L^1(\mathbb{R}^n)} \le \prod_{i=1}^d \|f_i\|_{L^{n-1}(\mathbb{R}^{n-1})}.$$

Proof of Lemma 4.11. The case $n = 2$ is clear: $f(x) = f_1(x_2)f_2(x_1)$ and

$$\int_{x_1,x_2} |f| = \left(\int_{x_2} |f_1|\right)\left(\int_{x_1} |f_2|\right).$$

Induction: Assume $n \geq 2$ is proved, then write $f = f_{n+1}(\tilde{x}_{n+1})F(x)$, $F(x) = f_1(\tilde{x}_1)\cdots f_n(\tilde{x}_n)$ and

$$\int_{x_1,\dots,x_n\in\mathbb{R}^n} |f(\cdot,x_{n+1})| \leq \|f_{n+1}\|_{L^n(\mathbb{R}^n)}\|F(\cdot,x_{n+1})\|_{L^{n/(n-1)}(\mathbb{R}^n)}.$$

Apply the induction assumption to $(x_1,\dots,x_n) \mapsto f_1^{n/(n-1)}(\cdot,x_{n+1})\cdots f_n^{n/(n-1)}(\cdot,x_{n+1})$:

$$\begin{aligned}\int_{x_1,\dots x_n} |f(\cdot,x_{n+1})| &\leq \|f_{n+1}\|_{L^n(\mathbb{R}^n)}\left(\prod_{i=1}^n \left\|f_i^{\frac{n}{n-1}}(\cdot,x_{n+1})\right\|_{L^{n-1}(\mathbb{R}^{n-1})}\right)^{\frac{n-1}{n}}\\ &= \|f_{n+1}\|_{L^n(\mathbb{R}^n)}\left(\prod_{i=1}^n \|f_i(\cdot,x_{n+1})\|_{L^n(\mathbb{R}^{n-1})}\right).\end{aligned}$$

Integate finally x_{n+1}:

$$\begin{aligned}\|f\|_{L^1(\mathbb{R}^{n+1})} &\leq \|f_{n+1}\|_{L^n(\mathbb{R}^n)}\int_{x_{n+1}}\left(\prod_{i=1}^n \|f_i(\cdot,x_{n+1})\|_{L^n(\mathbb{R}^{n-1})}\right)\mathrm{d}x_{n+1}\\ &\leq \|f_{n+1}\|_{L^n(\mathbb{R}^n)}\prod_{i=1}^n\left(\int_{x_{n+1}}\|f_i(\cdot,x_{n+1})\|^n_{L^n(\mathbb{R}^{n-1})}\,\mathrm{d}x_{n+1}\right)^{\frac{1}{n}}\\ &\leq \|f_{n+1}\|_{L^n(\mathbb{R}^n)}\prod_{i=1}^d \|f_i\|_{L^n(\mathbb{R}^n)}\end{aligned}$$

which proves the case $n+1$ and concludes the proof. □

Let us go back to the proof of the proposition with the lemma at hand. Consider $u \in C_c^\infty(\mathbb{R})$ and argue again by density. Define $v := |u|^{t-1}u$ with $\partial v = t|u|^{t-1}\partial u$, for any partial derivative ∂. Compute on v for any $1 \leq i \leq n$:

$$\begin{aligned}|v(x)| &\leq \left|\int_{-\infty}^{x_i}\frac{\partial v}{\partial x_i}(x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_n)\,\mathrm{d}y\right|\\ &\leq \int_{-\infty}^{+\infty}\left|\frac{\partial v}{\partial x_i}(x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_n)\right|\mathrm{d}y =: f_i(\tilde{x}_i).\end{aligned}$$

It implies by symmetry $|v|^{d/(d-1)} \leq \prod_{i=1}^d f_i^{1/(d-1)}$ and one can apply the lemma:

$$\|v\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \leq \left(\prod_{i=1}^d\left\|f_i^{\frac{1}{d-1}}\right\|_{L^{d-1}(\mathbb{R}^{d-1})}\right)^{\frac{d-1}{d}} \leq \left(\prod_{i=1}^d\|f_i\|^{\frac{1}{d-1}}_{L^1(\mathbb{R}^{d-1})}\right)^{\frac{d-1}{d}} \leq \prod_{i=1}^d\left\|\frac{\partial v}{\partial x_i}\right\|^{\frac{1}{d}}_{L^1(\mathbb{R}^d)}$$

which implies

$$\|u\|^t_{L^{\frac{td}{d-1}}(\mathbb{R}^d)} \leq t\prod_{i=1}^d\left\||u|^{t-1}\frac{\partial u}{\partial x_i}\right\|^{\frac{1}{d}}_{L^1(\mathbb{R}^d)} \leq t\prod_{i=1}^d\left(\|u\|^{t-1}_{L^{p'(t-1)}(\mathbb{R}^d)}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\mathbb{R}^d)}\right)^{\frac{1}{d}} \leq t\|u\|^{t-1}_{L^{p'(t-1)}(\mathbb{R}^d)}\|\nabla u\|_{L^p(\mathbb{R}^d)}$$

where $p' := p/(p-1)$. We then choose (in a unique way) t so that the exponents match: $td/(d-1) = p'(t-1) = p^*$, which gives the result of the statement. □

Proposition 4.12. *Assume $d = p$. We have $W^{1,p}(\mathbb{R}^d) \subset L^q(\mathbb{R})$ for any $q \in [p,+\infty)$ and for any such q there is $C_q > 0$ so that*

$$\forall\, u \in W^{1,p}(\mathbb{R}^d), \quad \|u\|_{L^q(\mathbb{R}^d)} \leq C_q\|u\|_{W^{1,p}(\mathbb{R}^d)}.$$

Proof of Proposition 4.12. We define $v := |u|^{t-1}u$ with $\partial v = t|u|^{t-1}\partial u$ for any partial derivative ∂. We perform the same calculation as above based on the Sobolev-Gagliardo-Nirenberg lemma to get:

$$\|u\|^t_{L^{\frac{td}{d-1}}(\mathbb{R}^d)} \leq t\|u\|^{t-1}_{L^{\frac{(t-1)d}{d-1}}(\mathbb{R}^d)} \|\nabla u\|_{L^d(\mathbb{R}^d)} \Longrightarrow \|u\|_{L^{\frac{td}{d-1}}(\mathbb{R}^d)} \leq C\left(\|u\|_{L^{\frac{(t-1)d}{d-1}}(\mathbb{R}^d)} + \|\nabla u\|_{L^d(\mathbb{R}^d)}\right).$$

By Hölder's and Mikowski's inequality it implies for any $t \geq d$:

$$\|u\|_{L^{\frac{td}{d-1}}(\mathbb{R}^d)} \leq C\left(\|u\|^\theta_{L^d(\mathbb{R}^d)}\|u\|^{1-\theta}_{L^{\frac{td}{d-1}}(\mathbb{R}^d)} + \|\nabla u\|_{L^d(\mathbb{R}^d)}\right) \leq \frac{1}{2}\|u\|_{L^{\frac{td}{d-1}}(\mathbb{R}^d)} + C'\left(\|u\|_{L^d(\mathbb{R}^d)} + \|\nabla u\|_{L^d(\mathbb{R}^d)}\right)$$

with $\theta := (d-1)/(t^2 - td + d - 1) \in (0, 1]$, which concludes the proof. ☐

Proposition 4.13. *Assume $d < p$. We have $W^{1,p}(\mathbb{R}^d) \subset C^{1-\frac{d}{p}}(\mathbb{R})$ and there is $C > 0$ so that*

$$\forall\, u \in W^{1,p}(\mathbb{R}^d), \quad \|u\|_{C^{1-\frac{d}{p}}(\mathbb{R}^d)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^d)}.$$

Proof of Proposition 4.13. We first prove the Hölder regularity. Consider the cube $Q = [-r, r]^d$. Call $\bar{u}$ the average of u on that cube. Then for any $y \in Q$:

$$\begin{aligned}
|\bar{u} - u(y)| &\leq \frac{1}{|Q|}\int_Q |u(x) - u(y)| \,\mathrm{d}x \\
&\leq \frac{r}{|Q|}\int_Q\int_0^1 \sum_{i=1}^d \left|\frac{\partial u}{\partial x_i}(y + z(x-y))\right| \mathrm{d}z\,\mathrm{d}x \\
&\leq \frac{Cr}{|Q|}\int_0^1 z^{-n}\left(\int_{(1-z)y+zQ} \sum_{i=1}^d \left|\frac{\partial u}{\partial x_i}(\tilde{x})\right| \mathrm{d}\tilde{x}\right) \mathrm{d}z \\
&\leq \frac{Cr}{|Q|}\int_0^1 z^{-n}\left(\sum_{i=1}^d \left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(Q)} \lambda(zQ)^{\frac{1}{p'}}\right) \mathrm{d}z \\
&\leq C'\|\nabla u\|_{L^p(Q)} r^{1-d+\frac{d}{p'}} \int_0^1 z^{-d+\frac{d}{p'}}\,\mathrm{d}z \leq Cr^{1-\frac{d}{p}}\|\nabla u\|_{L^p(Q)}.
\end{aligned}$$

This proves the Hölder regularity with index $1 - d/p$ by triangular inequality. Finally the L^∞ control is obtained as follows: any point $x \in \mathbb{R}^d$ belongs to a cube Q as above and $|u(x)| \lesssim |\bar{u}| + Cr^{1-d/p}\|\nabla u\|_{L^p(Q)} \leq C'(\|u\|_{L^p(Q)} + \|\nabla u\|_{L^p(Q)})$ for some constants $C, C' > 0$. ☐

Proof of Theorem 4.8. We sketch the argument (a more technical discussion will be given in the example sheet). Observe that as long as $p < d$ we continue applying the first proposition, which results into the loss of one derivative and the Lebesgue exponent p increasing by the transformation $\varphi(p) = pd/(d-p) > p$. This transformation maps $[d/(k+1), d/k)$ to $[d/k, d/(k-1))$ for $k \geq 2$ and $[d/2, d)$ to $[d, +\infty)$, with $d/(k+1)$ mapped to d/k. Therefore given $p \in [1, +\infty)$, (1) the case $s \leq d/p$ is proved by the previous propositions, (2) when $s > d/p$ the number of necessary iteration to increase the integrability exponent beyond d is s so that $p > d/s$, and we conclude by applying the third proposition once $p > d$. ☐

4.3. Sobolev spaces on an open set.

Definition 4.14. *We consider $\mathcal{U}$ a bounded and open set of $\mathbb{R}^d$ with smooth boundary $\partial\mathcal{U}$. We define the Sobolev space $W^{s,p}(\mathcal{U})$ on $\mathcal{U}$, for $s \in \mathbb{N}$ and $p \in [1 + \infty)$, as a subset of $L^p(\mathcal{U})$ by building the completion of the vector space $C^\infty(\mathbb{R}^d)$ (infinitely differentiable) endowed with the norm $\|g\|_{W^{s,p}(\mathcal{U})} := \left(\sum_{|\alpha|\leq s}\|\partial^\alpha_x g\|^p_{L^p(\mathcal{U})}\right)^{\frac{1}{p}}$. It means $W^{s,p}(\mathcal{U}) = \overline{C^\infty(\mathcal{U})}^{\|\cdot\|_{W^{s,p}(\mathbb{R}^d)}} \subset L^p(\mathcal{U})$. We write $H^s(\mathcal{U}) = W^{s,2}(\mathcal{U})$ in the case $p = 2$. We also define the Sobolev space $W^{s,p}_0(\mathcal{U})$ as a subset of $L^p(\mathcal{U})$ by building the completion of the vector space $C^\infty_c(\mathbb{R}^d)$ (infinitely differentiable with compact support included in $\mathcal{U}$) endowed with the same norm $W^{s,p}(\mathcal{U})$. It means $W^{s,p}_0(\mathcal{U}) = \overline{C^\infty_c(\mathcal{U})}^{\|\cdot\|_{W^{s,p}(\mathbb{R}^d)}} \subset L^p(\mathcal{U})$. We write $H^s_0(\mathcal{U}) = W^{s,2}_0(\mathcal{U})$ in the case $p = 2$.*

Remark 4.15. *As before the space $W^{s,p}(\mathcal{U})$ can be defined equivalently through the existence of generalised derivatives in $L^p(\mathcal{U})$ up to order s (including the case $p = +\infty$). A direct definition (without density) of $W_0^{s,p}(\mathcal{U})$ would however require* trace conditions.

Exercise 52. *Show that $H_0^1(\mathcal{U}) \subset H^1(\mathcal{U}) \subset L^2(\mathcal{U})$ are Hilbert spaces. Show also that for any $u, v \in H_0^1(\mathcal{U})$ and any first-order partial derivative ∂ we have $\int_{\mathcal{U}} (\partial u) v \, \mathrm{d}x = -\int_{\mathcal{U}} u(\partial v) \, \mathrm{d}x$. (Actually check that it is enough that only one of the two functions u and v is in $H_0^1(\mathcal{U})$, while the other one can be merely in $H^1(\mathcal{U})$.)*

Theorem 4.16. *The Sobolev inequalities extend to the case of a smooth bounded domain $\mathcal{U}$, or a half-plane.*

Proof. We only have time to sketch the proof. It is based on the construction of an *extension operator* $P : W^{1,p}(\mathcal{U}) \to W^{1,p}(\mathbb{R}^d)$ s.t. there is $C > 0$ s.t. for all $f \in W^{1,p}(\mathcal{U})$ it holds $Pf_{|\mathcal{U}} = f$ and $\|Pf\|_{L^p(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathcal{U})}$, $\|Pf\|_{W^{1,p}(\mathbb{R}^d)} \le C\|f\|_{W^{1,p}(\mathcal{U})}$. This operation is constructed by *reflexion* on a half-space, and then using a partition of the unit and mapping local neighborhoods of the boundary to ones where the boundary is flat. □

4.4. **Distributions.** The Sobolev spaces are particular subspaces of the larger "universal" space of *distributions* or "generalized functions" (theory of L. Schwartz [6, 7]).

Denote $\mathcal{D} := C_c^\infty(\mathbb{R}^d)$ and $\mathcal{D}'$ its dual (the space of continuous linear forms). Then $\mathcal{D}'$ is the space of distributions: locally integrable functions embed into it but also measures like the Dirac distribution or even derivation of measures like the dipole. And the *distributional derivative* always makes sense in this space: $\langle D^\alpha f, \varphi \rangle_{(\mathcal{D}',\mathcal{D})} = (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle_{(\mathcal{D}',\mathcal{D})}$.

The topology on $\mathcal{D}$ is given by an *inductive limit*: for a sequence of compact sets $K_i \to \mathbb{R}^d$ one can define a topology on $\mathcal{D}(K_i)$ (smooth functions with support included in K_i) by the family of semi-norms $\max_{K_i} |\partial^\alpha \varphi|$ for all $\alpha \in \mathbb{N}^d$. Then the topology on $\mathcal{D}$ is the *final topology* for the family of inclusions maps $\mathcal{D}(K_i) \mapsto \mathcal{D}$ (finest topology making all these maps continuous). This results in a topology with no countable basis of neighborhoods, however most sequential convergence-boundedness-compactness results are still true due to the particular structure of this inductive limit. Moreover one rarely works in such a general space, but rather in smaller subspaces (like Sobolev spaces) with more structure, and inspired by the equation at hand.

A smaller space of ("tempered") distribution $\mathcal{S}'$ is given as the dual of $\mathcal{S}$ that we have already encountered. It is endowed with a metric considering the decay at all order of all derivatives. It is a convenient space of distributions for which the Fourier transform naturally extends by duality.

4.5. **The Dirichlet problem for the Poisson equation.** One of the oldest PDE problems is the so-called *Dirichlet problem*: solving $\Delta f = g$ for f on an open set $\mathcal{U}$ with some boundary conditions $f = h$ on $\partial\mathcal{U}$. It corresponds to the distribution of temperatures at equilibrium under Fourier's law for instance. Let us first consider the simplest case, when the prescribed boundary values are assumed to vanish. Consider *a priori* some f which satisfies $\Delta f(x) = g(x)$ for any $x \in \mathcal{U}$ and $u \in C^2(\mathcal{U}) \cap C^1(\overline{\mathcal{U}})$ and $u(x) = 0$ on $x \in \partial\mathcal{U}$.

4.5.1. *The key a priori estimate.* An important idea in PDE is that of *a priori estimates*, i.e. searching a priori necessary estimates valid for smooth solutions, assuming their existence. Hence assume $f \in C^2(\mathcal{U}) \cap C^1(\overline{\mathcal{U}})$ is solution with $f = 0$ on $\partial\mathcal{U}$. Since we do not prescribe anything on the first derivative, we need to establish en estimate that does not depend boundary integrals of the gradient on $\partial\mathcal{U}$. We multiply the equation by u and integrate to obtain

$$\int_{\mathcal{U}} (\Delta f) f \, \mathrm{d}x = \int_{\mathcal{U}} g f \, \mathrm{d}x.$$

Integrating by parts (in view of the boundary conditions[7]), we get

$$\int_{\mathcal{U}} |\nabla f|^2 \, \mathrm{d}x = -\int_{\mathcal{U}} g f \, \mathrm{d}x \le \|g\|_{L^2(\mathcal{U})} \|f\|_{L^2(\mathcal{U})}.$$

[7] Note in particular why $C^1(\overline{\mathcal{U}})$ is natural.

In order the relate the LHS and RHS norms of f we prove now the following important result:

Theorem 4.17 (Poincaré's inequality with Dirichlet conditions). *Let $\mathcal{U} \subset \mathbb{R}^\ell$ be a open bounded set such that $\partial\mathcal{U}$ is smooth. Then there exists $C_\mathcal{U} > 0$ (only depending on $\mathcal{U}$) such that the following holds. Let $f \in C^1(\overline{\mathcal{U}})$ such that $f = 0$ on $\partial\mathcal{U}$, then*

$$\int_\mathcal{U} f(x)^2 \,\mathrm{d}x \le C_\mathcal{U} \int_\mathcal{U} |\nabla f(x)|^2 \,\mathrm{d}x.$$

The inequality extends to $f \in H^1_0(\mathcal{U})$ by density.

Proof of Theorem 4.17. Let us first an explicit intuitive proof when f is C^1. Consider any point $x = (x_1, \dots, x_d)$ in $\mathcal{U}$, and let $\bar{x}_1$ be such that $\bar{x} = (\bar{x}_1, x_2, \dots, x_d) \in \partial\mathcal{U}$. Then write

$$\int_\mathcal{U} f(x)^2 \,\mathrm{d}x = \int_\mathcal{U} \left(\int_{\bar{x}_1}^{x_1} \partial_{x_1} f(y, x_2, \dots, x_\ell) \,\mathrm{d}y \right)^2 \mathrm{d}x \le C \int_\mathcal{U} |\partial_{x_1} f(x)|^2 \,\mathrm{d}x \le C \int_\mathcal{U} |\nabla f(x)|^2 \,\mathrm{d}x$$

which concludes the proof. Note that C is the length of the greatest interval along the first axis included in $\mathcal{U}$. It is clear that this argument remains true under the more general condition that $\mathcal{U}$ is bounded along one of its direction only. It is also possible to prove this theorem in $H^1_0(\mathbb{R}^d)$ thanks to the Sobolev inequality applied for functions in $C^\infty_c(\mathcal{U})$. □

Remark 4.18. *Note that there are also* Poincaré's inequalities *in the whole space, provided the reference measure γ has some strong decay (essentially at least exponential) and regularity properties. The most famous example is the gaussian case $\gamma(x) = e^{-|x|^2}$:*

$$\left(\int_{\mathbb{R}^\ell} \left| f(x) - \int_{\mathbb{R}^\ell} f(y)\gamma(y) \,\mathrm{d}y \right|^2 \gamma(x) \,\mathrm{d}x \right)^{1/2} \le C_\gamma \left(\int_{\mathbb{R}^\ell} |\nabla f(x)|^2 \gamma(x) \,\mathrm{d}x \right)^{1/2}.$$

The proof is more involved than the one above, see for instance the 2011 exam paper of the course on kinetic theory for intermediate steps.

Apply Theorem 4.17 to deduce $\|f\|^2_{L^2(\mathcal{U})} \le C_\mathcal{U} \int_\mathcal{U} |\nabla f|^2 \,\mathrm{d}x \le C_\mathcal{U} \|g\|_{L^2(\mathcal{U})} \|f\|_{L^2(\mathcal{U})}$, which implies $\|u\|_{L^2(\mathcal{U})} \le C_\mathcal{U} \|f\|_{L^2(\mathcal{U})}$. Then by boostraping the information on the L^2 norm of f into the first a priori estimate we obtain $\int_\mathcal{U} |\nabla f|^2 \,\mathrm{d}x \le \|f\|_{L^2(\mathcal{U})} \|g\|_{L^2(\mathcal{U})} \le C_\mathcal{U} \|g\|^2_{L^2(\mathcal{U})}$. Combining the two last inequalities we can write $\|f\|^2_{H^1(\mathcal{U})} \le (C^2_\mathcal{U} + C_\mathcal{U}) \|g\|^2_{L^2(\mathcal{U})}$. Hence we have proved

Proposition 4.19. *Suppose $f_1, f_2 \in C^2(\mathcal{U}) \cap C^1(\overline{\mathcal{U}})$ satisfy $\Delta f_1 = \Delta f_2 = g$ on $\mathcal{U}$ with $f_1 = f_2 = h$ on $\partial\mathcal{U}$. Then $f_1 = f_2$.*

Proof of Proposition 4.19. The proof follows from the previous estimate applied to the solution $f = f_1 - f_2$ which solves $\Delta f = 0$ on $\mathcal{U}$ with $f = 0$ on $\partial\mathcal{U}$: $\|f\|^2_{H^1(\mathcal{U})} \lesssim \|0\|^2_{L^2(\mathcal{U})} = 0$ from our previous estimate, which shows by continuity that $f = 0$ everywhere. □

This last proposition solves the problem of uniqueness, but leaves open that of existence and continuity according to the data, which are the object of the next subsections.

4.6. **Existence of weak (generalised) solutions.** Weak formulations are an important tool for the analysis of PDEs that permit the transfer of concepts of *linear algebra* to solve the problems. In a weak formulation, an equation is no longer required to hold in the classical sense and has instead weak solutions only with respect to certain "test functions". This is equivalent to formulating the problem to require a solution in the sense of distributions. We introduce a formulation for weak solutions for the Poisson equation and show how to construct solutions using Riesz representation Theorem. (An easy generalisation to slightly more general elliptic problems is provided by the *Lax-Milgram theorem.*)

Let us define the notion of weak solutions. Assume that $\Delta f = g$ with $f \in C^2(\mathcal{U}) \cap C^1(\overline{\mathcal{U}})$ and $f = 0$ on $\partial\mathcal{U}$ then for any $\varphi \in C^2(\mathcal{U}) \cap C^1(\overline{\mathcal{U}})$ with $\varphi = 0$ on $\partial\mathcal{U}$ we have

$$\langle\langle f, \varphi \rangle\rangle := \int_\mathcal{U} \nabla f \cdot \nabla\varphi \,\mathrm{d}x = -\int_\mathcal{U} (\Delta f)\varphi \,\mathrm{d}x = -\int_\mathcal{U} g\varphi \,\mathrm{d}x$$

where we have denoted by $\langle\langle\cdot,\cdot\rangle\rangle$ the bilinear symmetric form obtained by integrating the gradients, keeping the notation $\langle\cdot,\cdot\rangle$ for the usual $L^2(\mathcal{U})$ scalar product. Observe crucially that the objects in the LHS and RHS of this statement still make sense as soon as $u, v \in H^1_0(\mathcal{U})$.

Definition 4.20. *We call* generalised (or weak) solution *a function $f \in H^1_0(\mathcal{U})$ such that for all $\varphi in H^1_0(\mathcal{U})$ it holds $\langle\langle f,\varphi\rangle\rangle = -\langle g,\varphi\rangle$. Equivalently $f \in H^1_0(\mathcal{U})$ and $\langle f, \Delta\varphi\rangle = \langle g,\varphi\rangle$ for all $\varphi \in C^\infty_c(\mathcal{U})$ smooth of compact support in $\mathcal{U}$. (This latter equality is the statement that f is a distributional solution of $\Delta f = g$.)*

Exercise 53. *Prove the equivalence in the definition.*

Remark 4.21. *Note the important idea behind this reformulation: the boundary conditions have been enforced-encoded* in the functional space *itself.*

We can now state and prove the existence theorem:

Theorem 4.22. *Let $\mathcal{U} \subset \mathbb{R}^\ell$ a bounded open set with smooth boundary, and $g \in L^2(\mathcal{U})$. Then there exists a unique $f \in H^1_0(\mathcal{U})$ weak solution of $\Delta f = g$, in the sense defined above.*

Proof of Theorem 4.22. The proof is a straightfoward application of the Riesz representation Theorem: we consider the following linear form on $H^1_0(\mathcal{U})$: $G(\varphi) := -\langle g,\varphi\rangle$ which is continuous by Cauchy-Schwarz and Poincaré's inequalities: $\|G(\varphi)\| = \|\langle g,\varphi\rangle\| \le \|g\|_{L^2(\mathcal{U})}\|\varphi\|_{L^2(\mathcal{U})} \lesssim \|g\|_{L^2(\mathcal{U})}\|\nabla\varphi\|_{L^2(\mathcal{U})}$. Then the Riesz representation theorem applied in the Hilbert space $H^1_0(\mathcal{U})$ endowed with the equivalent norm $\sqrt{\langle\langle\cdot,\cdot\rangle\rangle}$, shows that there is a unique $f \in H^1_0(\mathcal{U})$ so that $G(\varphi) = \langle\langle f,\varphi\rangle\rangle$ for any $\varphi \in H^1_0(\mathbb{R}^d)$. which concludes the proof. □

Remark 4.23. *Observe moreover that in the previous statement the solution map $\mathfrak{S} : g \mapsto f$ is continuous from $L^2(\mathcal{U})$ to $H^1_0(\mathcal{U})$ since $\|G\|_{H^1_0(\mathcal{U})^*} = \sup_{\|\nabla\varphi\|_{L^2(\mathcal{U})}=1} |G(\varphi)| \lesssim \|g\|_{L^2(\mathcal{U})}$ and $G \mapsto f$ is an isometry in the representation theorem. In fact prove that $\mathfrak{S}$ is even continuous from $H^{-1}_0(\mathcal{U})$, the dual of $H^1_0(\mathcal{U})$ for the $L^2(\mathcal{U})$ scalar product $\langle\cdot,\cdot\rangle$, to $H^1_0(\mathcal{U})$: $\|f\|_{H^{-1}_0(\mathcal{U})} := \sup_{\|v\|_{H^1_0(\mathcal{U})}=1}\langle f,v\rangle_{L^2(\mathcal{U})}$.*

Once the solution is built, and assuming that $g \in C^\infty$, one can prove the regularity by using a priori estimates and Sobolev embeddings.

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