3. Fourier decomposition of functions

3.1. The Fourier transform.

Definition 3.1 (Fourier Transform on $L^1(\mathbb{R}^d)$). *Given $f \in L^1(\mathbb{R}^d)$ define its **Fourier transform** $\mathcal{F}(f)(\xi) := \int_{\mathbb{R}^d} e^{-2i\pi x\cdot\xi} f(x)\,\mathrm{d}x$ for all $\xi \in \mathbb{R}^d$. $\mathcal{F}$ is linear and bounded from $L^1(\mathbb{R}^d)$ to $(C_b(\mathbb{R}^d), \|\cdot\|_\infty)$.*

Proof. The application is well-defined since $|e^{-2i\pi x\cdot\xi} f(x)| = |f(x)|$ is integrable. Supremum bound on $\mathcal{F}(f)$ follows from Hölder's inequality and give $\|\mathcal{F}(f)\|_{L^\infty} \le \|f\|_{L^1}$. Continuity of $\xi \mapsto \mathcal{F}(f)(\xi)$ follows from the dominated convergence theorem. □

Proposition 3.2. *Basic properties on the Fourier transform:*
*(i) **Scaling**: given $f \in L^1(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}\setminus\{0\}$, $\mathcal{F}(f(\lambda\cdot))(\xi) = |\lambda|^{-d}\mathcal{F}(f)(\xi/\lambda)$.*
*(ii) **Duality convolution/multiplication**: given $f, g \in L^1(\mathbb{R}^d)$, the convolution $f * g \in L^1(\mathbb{R}^d)$ and $\mathcal{F}(f*g) = \mathcal{F}(f)\mathcal{F}(g)$.*
*(iii) **Duality translation/phase change**: given $f \in L^1(\mathbb{R})$ and $h \in \mathbb{R}^d$, define $\tau_h f := f(\cdot - h) \in L^1(\mathbb{R}^d)$ then $\mathcal{F}(\tau_h f)(\xi) = e^{2i\pi h\cdot\xi}\mathcal{F}(f)(\xi)$.*
*(iv) **Duality regularity/decay**: (1) if $f \in L^1(\mathbb{R}^d)$ and $x_j f \in L^1(\mathbb{R}^d)$ then $\mathcal{F}(f)$ is differentiable wrt ξ_j and $\partial_{\xi_j}\mathcal{F}(f) = -2i\pi\mathcal{F}(x_j f)$; (2) if $f \in L^1(\mathbb{R}^d)$ is C^1 with $\partial_{x_j} f \in L^1(\mathbb{R}^d)$ then $\mathcal{F}(\partial_{x_j} f)(\xi) = 2i\pi\xi_j\mathcal{F}(f)(\xi)$.*

Remark 3.3. *Last point (iv) is also a duality between derivation and multiplication, and motivates Fourier theory for reducing differential equations to algebraic equations. We have used in point (ii) the* Young inequality*: given $p, q, r \in [1, +\infty]$ s.t. $1/p + 1/q = 1/r + 1$ and $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R}^d)$, the function $f * g \in L^r(\mathbb{R}^d)$ with $\|f*g\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q}$ (prove it with Hölder's inequality).*

Proof. Points (i) and (iii) follow by change of variable, point (ii) follow by Fubini's theorem. To prove (iv)-(1), observe that $\xi_j \mapsto e^{-2i\pi x\cdot\xi} f(x)$ is differentiable for a.e $x \in \mathbb{R}^d$, and the derivative has modulus less than $2\pi|x_j f| \in L^1(\mathbb{R}^d)$, hence the mean-value and dominated convergence theorem show that the derivative of $\xi_j \mapsto \mathcal{F}(f)(\xi)$ exists and is $\partial_{\xi_j}\mathcal{F}(f) = -2i\pi\mathcal{F}(x_j f)$. To prove (iv)-(2) one uses integration by part, the only thing to be careful with is to take a limit $[-M, M]^d \xrightarrow{M\to\infty} (-\infty, +\infty)^d$ for the domain of integration with $|f| \to 0$ on the boundary. □

Proposition 3.4 (Oscillation and decay). *Given $f \in L^1(\mathbb{R}^d)$, the Fourier transform $\mathcal{F}(f)(\xi) \xrightarrow{\xi\to\infty} 0$.*

Remark 3.5. *It is a particular case of application of the more general Riemann-Lebesgue lemma.*

Proof. We have proved that $C_c^\infty(\mathbb{R}^d)$ is dense in $L^1(\mathbb{R}^d)$. Assume $f \in C_c^\infty(\mathbb{R}^d)$, then define $\Delta := \partial^2_{x_1} + \cdots + \partial^2_{x_d}$ and since $\partial_{x_i} f, \partial^2_{x_i} f \in L^1(\mathbb{R}^d)$ from the previous proposition $(1-\Delta)f \in L^1(\mathbb{R}^d)$ and $\mathcal{F}((1-\Delta)f)(\xi) = (1 + 4\pi^2|\xi|^2)\mathcal{F}(f)(\xi)$. Therefore since $\mathcal{F}((1-\Delta)f)(\xi)$ is bounded, the initial Fourier transform $|\mathcal{F}(f)(\xi)| \le C(1+4\pi^2|\xi|^2)^{-1}$ and goes to zero as $\xi \to \infty$. Now for $f \in L^1(\mathbb{R}^d)$ and any $\varepsilon > 0$, take $g \in C_c^\infty(\mathbb{R}^d)$ s.t. $\|f - g\|_{L^1} \le \varepsilon/2$ and use $\sup_{|\xi|\ge M} |\mathcal{F}(g)(\xi)| \le \varepsilon/2$ for M large enough from above. Deduce $\sup_{|\xi|\ge M} |\mathcal{F}(f)(\xi)| \le \sup_{|\xi|\ge M} |\mathcal{F}(g)(\xi)| + \|f-g\|_{L^1(\mathbb{R}^d)} \le \varepsilon$. □

Remark 3.6. *The map $\mathcal{F} : L^1(\mathbb{R}^d) \to C_0(\mathbb{R}^d)$ continuously where $C_0(\mathbb{R}^d)$ denotes the continuous functions going to zero at infinity (endowed with the supremum norm). Prove that is* not *surjective.*

Proposition 3.7 ("Fixed point when $a = \pi$"). *Given $a > 0$: $\mathcal{F}(e^{-ax^2})(\xi) = \sqrt{\frac{\pi}{a}} e^{-\frac{\pi^2}{a}\xi^2}$ in $\mathbb{R}$.*

Proof. Check that $f(x) = e^{-ax^2}$ on $\mathbb{R}$ satisfies $f'(x) - = -2axf(x)$ and use to establish the differential equation $g'(\xi) = -2\pi^2 a^{-1} g(\xi)$ on $g(\xi) := \mathcal{F}(f)(\xi)$ and solve it. □

Theorem 3.8 (Inversion in $L^1(\mathbb{R}^d)$). *Consider $f \in L^1(\mathbb{R}^d)$ s.t. $\mathcal{F}(f) \in L^1(\mathbb{R}^d)$ then $g(y) := \int_{\mathbb{R}^d} e^{2i\pi y\cdot\xi}\mathcal{F}(f)(\xi)\,\mathrm{d}\xi$ is equal to $f(y)$ almost everywhere, i.e. $\overline{\mathcal{F}} \circ \mathcal{F}(f) = f$ or $\mathcal{F}^2(f) = \check{f} := f(-\cdot)$.*

Proof. This relies on an important calculation: define $H(\xi) := e^{-2\pi(\sum|\xi_j|)}$ and $h_k(y) := \int H(\xi/k) e^{2i\pi y\cdot\xi}\,\mathrm{d}\xi$. Prove by explicit calculation that $\int H = \pi^{-d}$ and $h_k \ge 0$ and $\int h_k = 1$ and moreover for any η,

$\int_{|y|\leq\eta} h_k \to 1$ as $k \to \infty$: the sequence h_k is an approximation of the unit. By Fubini's theorem $f * h_k(y) = \int H(\xi/k)e^{2i\pi y\cdot\xi}\mathcal{F}(f)(\xi)\,\mathrm{d}\xi$ and use here $\mathcal{F}(f) \in L^1(\mathbb{R}^d)$ to pass to the limit as $k \to \infty$. □

Corollary 3.9. *The map $\mathcal{F} : L^1(\mathbb{R}^d) \to C_0(\mathbb{R}^d)$ is injective.*

Proof. By linearity consider $f \in L^1(\mathbb{R}^d)$ s.t. $\mathcal{F}(f) = 0 \in L^1(\mathbb{R}^d)$ and apply the previous inversion result to get $f = \overline{\mathcal{F}}(0) = 0$. □

Theorem 3.10 (Plancherel)**.** *Given $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ then the Fourier transform $\mathcal{F}(f) \in L^2(\mathbb{R}^d)$ and $\|\mathcal{F}(f)\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}$.*

Proof. Define $\tilde{f}(x) := \overline{f(-x)}$. Since $f \in L^1 \cap L^2$ check that $\tilde{f} \in L^1 \cap L^2$ by change of variable, and $g := f * \tilde{f} \in L^1 \cap L^\infty$; moveover it is uniformly continuous: for $h \in \mathbb{R}^d \setminus \{0\}$, $\|\tau_h g - g\|_{L^\infty} \leq \|\tau_h f - f\|_{L^2}\|f\|_{L^2} \to 0$ as $h \to 0$ (continuity of the translation in L^2). Hence $g * h_k \to g$ uniformly as $k \to \infty$, and thus $(g * h_k)(0) \to g(0) = \|f\|^2_{L^2}$. But $(g * h_k)(0) = \int H(\xi/k)\mathcal{F}(g)(\xi)\,\mathrm{d}\xi$ and from property (ii) above $\mathcal{F}(g) = |\mathcal{F}(f)|^2$. Finally $H(\xi/k) \geq 0$ and converges increasingly to 1 hence (monotone convergence) $(g * h_k)(0) \to \int |\mathcal{F}(f)|^2$ and $\mathcal{F}(f) \in L^2$ with the equality of L^2 norms. □

Corollary 3.11. *There is unique a isometry from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ denoted $f \mapsto \hat{f}$ s.t. for any $f \in L^1 \cap L^2$, $\hat{f} = \mathcal{F}(f)$, we call it* Fourier-Plancherel transform.

Proof. The application $\mathcal{F} : L^1 \cap L^2 \to L^2$ is linear isometric for the L^2 norm, hence uniformly continuous. Since $C_c^\infty \subset L^1 \cap L^2$ and C_c^∞ dense in L^2, $L^1 \cap L^2$ is also dense in L^2. Finally L^2 is complete normed space, hence there is a unique continuous extension to L^2, it is isometric by limit. □

Remark 3.12. *Be careful that the usual formula defining $\mathcal{F}$ in L^1 does not always make sense in L^2 and results must be proved by density. Prove for instance by density that $\hat{\check{f}} = \check{\hat{f}}$.*

Proposition 3.13 (Inversion in $L^2(\mathbb{R}^d)$)**.** *The Fourier-Plancherel transform is a bijective isometry $\hat{\cdot} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ and $\hat{\hat{f}} = \check{f}$.*

Proof. First consider $f \in L^1 \cap L^2$ and the approximation of the unit (for instance) $h_k \in L^1 \cap L^2$ constructed above. Then $f * h_k \in L^1 \cap L^2$ and $\mathcal{F}(f * h_k) = \mathcal{F}(f)\mathcal{F}(h_k) = (L^2)(L^2) \in L^1$. Hence by inversion in L^1: $\mathcal{F}\mathcal{F}(f * h_k) = (f * h_k)\check{}$. But $\mathcal{F}\mathcal{F}(f * h_k) = (f * h_k)\hat{}\hat{}$ and passing to the limit gives by convergence in L^2 (using the isometry property) $\hat{\hat{f}} = \check{f}$. Finally both $\hat{}$ and $\check{}$ are isometric hence continuous and the result extends to L^2 by density. □

Finally it is natural to search for another "fixed space" instead of L^2 that has regularity and decay.

Proposition 3.14 (The Schwartz space)**.** *The Schwartz space or space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^d)$ on $\mathbb{R}^d$ is the space of smooth functions s.t. all their derivatives decay faster than any polynomial at infinity: $f \in \mathcal{S}(\mathbb{R}^d)$ iff $\forall\, \alpha \in \mathbb{N}^d$, $\beta \in \mathbb{N}$, $|x|^\beta \partial^\alpha f(x) \to 0$ as $|x| \to +\infty$. The Fourier transform $\mathcal{F}$ maps $\mathcal{S}(\mathbb{R}^d)$ to itself bijectively.*

Proof. The Fourier transform is well-defined on $\mathcal{S}$ since $\mathcal{S} \subset L^1$, check $\mathcal{F}(\mathcal{S}) \subset \mathcal{S}$ thanks to the duality regularity/decay and the bound on $\mathcal{F} : L^1 \to L^\infty$. The inversion then implies bijectivity. □

3.2. **Fourier series and orthonormal basis.** The Fourier transform decomposes a function on $\mathbb{R}^d$ into its "oscillatory modes", which form a continuum. On a bounded domain the corresponding decomposition involves "harmonics", i.e. discrete sets of possible frequencies[5].

Definition 3.15. *Consider a Hilbert space[6] H. A family $(f_i)_{i\in I}$ of elements of H (not necessarily countable) is said* orthonormal *if $\langle f_i, f_j\rangle = \delta_{ij}$ for any $i, j \in I$ ($\delta_{ij} = 1$ if $i = j$ and zero otherwise).*

[5] This relates to *continuum spectrum* vs. *discrete spectrum*.

[6] A pre-Hilbertian space would be enough here in fact.

Proposition 3.16. *There is a unique isometric linear application $I : \ell^2(I) \to H$ s.t. for any $\phi \in \ell^2(I)$ with finite support, $I(\phi) = \sum_{i\in I} \phi(i) f_i \in H$. Its range is $F := \overline{\mathrm{Span}(f_i, i \in I)}$ (closure of the vector subspace spanned by the family) and its inverse on it is given by the application $\hat{}: F \to \ell^2(I)$ s.t. $\hat f(i) = \langle f_i, f\rangle$ for any $i \in I$. Finally $I(\phi)\hat{} = \phi$ on $\phi \in \ell^2(I)$ and $I(\hat f) = P_F(f)$ on $f \in H$ where P_F is the orthogonal projection on F (hence $\|\hat f\|_{\ell^2(I)} \le \|f\|_H$).*

Remark 3.17. *We only consider countable orthonormal families for the exam. We however mention that when I is uncountable the space $\ell^2(I)$ is the space of functions $f : I \to \mathbb{C}$ s.t. f is non-zero only for a countable subset of I, with square-summability on this subset. Note also that P_F is the Hilbertian projection on a non-empty convex closed set that we have seen in the previous chapter.*

Proof. (We restrict to the case where I is countable) Observe that Pythagoras's theorem implies that an orthogonal family is linearly independent and that I is isometric on the set $D \subset \ell^2(I)$ of functions $f : I \to \mathbb{C}$ with finite support, i.e. non-zero only on a finite subset of I, and that $\hat{}$ is isometric on $I(\ell^2(I))$. Moreover D is dense in $\ell^2(I)$ by considering the limits of partial sums. There is thus a unique isometric extension I on $\ell^2(I)$. The identity $I(\phi)\hat{} = \phi$ is satisfied on $\phi \in D \subset \ell^2(I)$ and thus extends to $\ell^2(I)$ by density-continuity. Since $I(D) \subset \mathrm{Span}(f_i, i \in I)$ and $I(\ell^2(I)) = \overline{I(D)}$ by density-continuity, we deduce that $I(\ell^2(I)) \subset F$ and the whole F is attained by explicit construction of a pre-image. Finally we have for any $f \in H$ the decomposition $f = P_F(f) + P_{F^\perp}(f)$ (with $P_{F^\perp} = 1 - P_F$) and since $\langle f, f_i\rangle = \langle P_F(f), f_i\rangle$ for any $i \in I$, $\hat f = P_F(f)\hat{}$ for $f \in H$ and thus $I(\hat f) = P_F(f)$. □

Definition 3.18. *Given H Hilbert space and an orthonormal family $(f_i)_{i\in I}$, this family is called* ***Hilbert basis*** *if $F := \overline{\mathrm{Span}(f_i, i \in I)} = H$. When so H is isometrically isomorphic to $\ell^2(I)$.*

Exercise 45. *Compare the definition of a Hilbert basis with a standard (algebraic basis) of a vector space. Prove with Zorn's lemma that every Hilbert space has a Hilbert basis. Prove (without Zorn's lemma) that a Hilbert space is* separable *iff it has a* countable *Hilbert basis. Prove that if a Hilbert space has infinite dimension any algebraic basis is uncountable.*

Proposition 3.19 (Practical criterion for a Hilbert basis)**.** *Given H Hilbert space, an orthonormal family $(f_i)_{i\in I}$ is a Hilbert basis iff one of the following holds: (i) $\mathrm{Span}(f_i, i \in I)^\perp = \{0\}$, (ii) the family is a maximal orthonormal system, (iii) for any $f \in H$, $\|f\|_H = \|\hat f\|_{\ell^2(I)}$, (iv) for any $f, g \in H$, $\langle f, g\rangle = \sum_{i\in I} \overline{\hat f}(i)\hat g(i)$.*

Proof. The point (i) follows from proving that the orthogonal $A^\perp := \{g \in H \mid \forall f \in A,\ \langle f, g\rangle = 0\}$ of a set $A \subset H$ is equal to $\overline{A}^\perp$ (orthogonal of the closure of the set), which follows from the continuity of the inner product. To prove points (ii) and (iii): if H strictly larger than F then $F^\perp \neq \{0\}$ and one can find $g \in F^\perp$ with norm 1 and add it to the family (f_i) which would not be maximal, and this g also satisfies $1 = \|g\|_H \neq \|\hat g\|_{\ell^2(I)} = 0$. Finally (iv) is equivalent to (iii) by relating the norms and inner products. □

Example 1. Fourier series. Consider the Hilbert space $H = L^2([0, 1])$.

Theorem 3.20 (Fejér)**.** *Given $f \in C^0([0, 1])$, the sequence $\frac{1}{N+1}\sum_{n=0}^N \left(\sum_{k=-n}^{k=n} \hat f(k) e^{2i\pi kx}\right)$ converges uniformly to f, where $\hat f(k) := \int_0^1 f(x) e^{-2i\pi kx}\,\mathrm{d}x$ denotes the k-th Fourier mode of f.*

Proof. Prove by direct calculation that

$$\frac{1}{N+1}\sum_{n=0}^N \left(\sum_{k=-n}^{k=n} \hat f(k) e^{-2i\pi kx}\right) = \frac{1}{2\pi}\int_0^{2\pi} f(x-y) K_N(y)\,\mathrm{d}y$$

with the *Fejér kernel*

$$K_N(y) := \frac{1}{N+1}\sum_{n=0}^N \sum_{k=-n}^{k=+n} e^{2i\pi kx} = \frac{1}{N+1}\frac{\sin^2(\pi(N+1)y)}{\sin^2(\pi y)}$$

(that is computed by summing the geometric series). This kernel is a continuous 1-periodic function on $\mathbb{R}$ that satisfies $K_N \geq 0$ and $\int_0^1 K_N = 1$ (it is easier to check this last property on the first formula for K_N). Moreover for any $\delta \in (0, 1/2)$ one has $\int_{|z|\geq\delta} K_N(z)\,\mathrm{d}z \to 0$ as $N \to +\infty$. This shows that K_N is an approximation of the unit and implies that $K_N * f$ converges uniformly to f as $N \to \infty$. □

Corollary 3.21. *(i) The countable family ($f_n := e^{2i\pi kx}$), $k \in \mathbb{Z}$ is a Hilbert basis with $\hat{f}(k) = \int_0^1 f(x)e^{-2i\pi kx}\,\mathrm{d}x$. (ii) Polynomials are dense in $C^0([0,1])$ (Weierstrass theorem in $\mathbb{R}$).*

Proof. (i) The previous theorem proves that trigonometrical polynomials are dense in $C^0([0,1])$ and since $C^0([0,1])$ is dense in $L^2([0,1])$ this proves that $\overline{\mathrm{Span}(f_k, k \in \mathbb{Z})} = L^2([0,1])$. (ii) Each f_n can be approximated by polynomials uniformly on $[0,1]$ by Taylor expansion. □

Remark 3.22. *Because of (i) one has (Plancherel) $\|f\|_{L^2([0,1])} = \sum_{k\in\mathbb{Z}} |\hat{f}(k)|^2$ and moreover f can be approximated by a finite number of its Fourier modes.* ***However*** *be careful that in general $f = \lim_{n\to+\infty} \sum_{k=-n}^{k=n} \hat{f}(n)e^{in\cdot}$ only holds in the sense of an L^2 convergence, not necessarily pointwise. The next theorem addresses this question.*

The next two theorems are non-examinable:

Theorem 3.23 (Dirichlet)**.** *Consider $f : \mathbb{R} \to \mathbb{C}$ measurable 1-periodic and integrable on $[0,1]$. At any point $x_0 \in \mathbb{R}$ where (1) f admits left limit $f(x_0^-)$ and right limit $f(x_0^+)$, (2) the functions $|f(x_0+t) - f(x_0^+)|/t$ and $|f(x_0-t) - f(x_0^-)|/t$ are integrable at 0^+, then the symmetric Fourier sum at x_0 converges as $\sum_{k=-n}^{n} \hat{f}(k)e^{-2i\pi kx_0} \xrightarrow{n\to+\infty} \frac{f(x_0^+)+f(x_0^-)}{2}$.*

Proof. The proof is based on writing the symmetric sum $S_n(f)(x) := \sum_{k=-n}^{n} \hat{f}(k)e^{-2i\pi kx}$ as a convolution $(D_n * f)(x)$ with the *Dirichlet kernel* $D_n(x) = \sum_{k=-n}^{n} e^{2i\pi kx} = \frac{\sin(\pi(2n+1)x)}{\sin(\pi x)}$, and then using a variant of the Riemann-Lebesgue lemma:

$$S_n(f)(x_0) - \frac{f(x_0^+) + f(x_0^-)}{2} = \int_0^{1/2} \sin(\pi(2n+1)y) \left[\frac{f(x_0+t) + f(x_0-t) - f(x_0^+) - f(x_0^-)}{\sin(\pi t)}\right] \mathrm{d}t$$

which goes to zero as $n \to \infty$ since the function under brackets is integrable. □

Theorem 3.24 (Poisson summation)**.** *Consider a continuous function $f : \mathbb{R} \to \mathbb{C}$ s.t. there are $C > 0$, $\alpha > 1$ with $|f(x)| \leq C(1+|x|)^{-\alpha}$ for all $x \in \mathbb{R}$, and with $\sum_{k\in\mathbb{Z}} |\mathcal{F}(f)(k)| < +\infty$. Then $\sum_{n\in\mathbb{Z}} f(n) = \sum_{k\in\mathbb{Z}} \mathcal{F}(f)(k)$ (equality of two absolutely convergent series).*

Proof. Consider the function $\varphi(t) := \sum_{n\in\mathbb{Z}} f(t+n)$ which is well-defined, 1-periodic, and C^1 using the uniform convergence of the series and of the series of derivatives. Its Fourier coefficients are $\hat{\varphi}(k) = \mathcal{F}(f)(k)$ (the switching of summation and integration to compute this is justified because of the bound we have on $|f(x)|$). Since the series of Fourier coefficients is absolutely converging the function $g(t) := \sum_{k\in\mathbb{Z}} \hat{\varphi}(k)e^{-2i\pi kt}$ is well-defined and continuous and 1-periodic on $\mathbb{R}$. The two functions φ and g have the same Fourier coefficients hence are equal as L^2 functions thus (continuity) everywhere, and the formula is given at $t = 0$. □

Exercise 46. *Use the Poisson summation formula to prove that (1) $\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, (2) $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^4} = \frac{7\pi^4}{720}$, (3) $\sum_{n=-\infty}^{+\infty} e^{-\pi n^2 s} = \frac{1}{\sqrt{s}} \sum_{n=-\infty}^{+\infty} e^{-\pi n^2 s^{-1}}$.*

Example 2. Hermite polynomials. Consider the following polynomials for $k \geq 0$ integer: $H_k(x) := (-1)^k e^{x^2/2} \frac{\mathrm{d}^k}{\mathrm{d}x^k} e^{-x^2/2}$ and $\tilde{H}_k(x) := (-1)^k e^{x^2} \frac{\mathrm{d}^k}{\mathrm{d}x^k} e^{-x^2}$. Prove that (1) $\tilde{H}_k(x) = 2^{k/2} H_k(x\sqrt{2})$, (2) H_k and $\tilde{H}_k$ are polynomials with degree k, (3) each of the families $(H_k)_{k\geq0}$ and $(\tilde{H}_k)_{k\geq0}$ is orthogonal in the Hilbert space $H = L^2(\mathbb{R}, \mathrm{d}\mu)$ with the measure $\mu = e^{-x^2/2}$. Find induction relations on these polynomials.

Example 3. Haar polynomials. Another interesting example will be considered more in details in the example sheet: the Haar basis which was the first example of *wavelets*: $H_1(x) := \chi_{[0,1]}(t)$ and for $k \geq 0$ and $1 \leq l \leq 2^k$: $H_{2^k+l}(x) := \chi_{\left[\frac{2l-2}{2^{k+2}}, \frac{2l-1}{2^{k+1}}\right]}(x) - \chi_{\left[\frac{2l-1}{2^{k+2}}, \frac{2l}{2^{k+1}}\right]}(x)$. The family $(H_m)_{m \geq 1}$ is a Hilbert basis of $L^2([0,1])$ and more generally a *Schauder basis* of $L^p([0,1])$ for $1 \leq p < +\infty$ (see the example sheet for the later).

3.3. **Solving equations with Fourier theory.** The Fourier theory was invented to reduce partial differential equations (PDEs) to algebraic equations or at least ordinary differential equations (ODEs). More precisely differential operators with constant coefficients are transformed into multiplication operators in the Fourier world. This motivates the following more general definition: given a differential operator P of order m: $Pf := \sum_{|\alpha| \leq m} a_\alpha(x) \partial_x^\alpha f$ with $a_\alpha(x)$ smooth functions, the *symbol* is defined as $\sigma(x,\xi) := \sum_{|\alpha| \leq m} a_\alpha(x) i^\alpha \xi^\alpha$ where $\xi^\alpha := \xi_1^{\alpha_1} \dots \xi_\ell^{\alpha_\ell}$ and the *principal symbol* is defined as $\sigma_p(x,\xi) := \sum_{|\alpha|=m} a_\alpha(x) \xi^\alpha$. The symbol corresponds to the action of the operator on plane waves $\sigma(x,\xi) = e^{-ix\cdot\xi} P(e^{ix\xi})$. The names elliptic-parabolic-hyperbolic are related to the conics described by the symbol. They relate however to much deeper mathematical and physical meaning. Here are a few standard examples:

3.3.1. *Elliptic equations.* Consider the *Laplace-Poisson equation*: $\Delta f = g$ on $\mathbb{R}^d$ for $g \in \mathcal{S}(\mathbb{R}^d)$. A solution $f \in \mathcal{S}(\mathbb{R}^d)$ would need to satisfy $\mathcal{F}(f)(\xi) = -(4\pi^2)^{-1}(\xi_1^2 + \xi_2^2 + \cdots + \xi_d^2)^{-1} \mathcal{F}(g)(\xi)$. Hence one can construct a solution in $\mathcal{S}(\mathbb{R}^d)$ with the help of the Fourier transform if $\mathcal{F}(g)$ vanishes like $\|\xi\|^2$ at $\xi = 0$. This is a sort of generalised zero-mass condition. The modified equation $(1-\Delta)f = g$ would not have this problem: in Fourier this new equation is $(4\pi^2)(1 + \xi_1^2 + \xi_2^2 + \cdots + \xi_d^2)\mathcal{F}(f)(\xi) = \mathcal{F}(g)(\xi)$ and the so-called *symbol* of the operator $(1 + \xi_1^2 + \xi_2^2 + \cdots + \xi_d^2)$ has no cancellation. The Laplace operator gave its name to a class of PDEs along the type of conic described by its symbol, the elliptic PDEs: no non-zero cancellation of the symbol, infinite regularisation, evolution problem overdetermined and unstable, Dirichlet problem...

3.3.2. *Hyperbolic equations.* Consider the *wave equation*: $\partial_t^2 f = \partial_x^2 f$ for $f = f(t,x)$ defined on $\mathbb{R}_+ \times \mathbb{R}$, and with $f(0,\cdot) = F$ and $\partial_t f(0,\cdot) = G$. Assuming that $F, G \in \mathcal{S}(\mathbb{R})$, one can construct a solution with the help of the Fourier transform in x by reducing this equation to a second-order ODE. The symbol agrees with the principal symbol, it is (denoting τ for the Fourier variable of t and ξ for the Fourier variable of x) $\sigma(t,x,\tau,\xi) = \sigma(\tau,\xi) = \tau^2 - \xi^2$, and the hyperbola conic described by the symbol gives its name to the class of equations that share key properties with the wave equation: "as many" cancellations of the symbol as possible while preserving causality, propagation of singularity, finite-speed of propagation of information (fluid mechanics, kinetic theory, general relativity...).

3.3.3. *Parabolic equations.* Consider the *heat equation*: $\partial_t f = \partial_x^2 f$ for $f = f(t,x)$ defined on $\mathbb{R}_+ \times \mathbb{R}$, and with $f(0,\cdot) = F \in \mathcal{S}(\mathbb{R})$. One can construct a solution with the help of the Fourier transform in x by reducing this equation to a first-order ODE. The symbol is $\sigma(t,x,\tau,\xi) = \sigma(\tau,\xi) = i\tau + \xi^2$ while the principal symbol is $\sigma_P(\tau,\xi) = \xi^2$. The parabola conic described by the symbol gives its name to the class of equations that share key properties with the heat equation: "degenerate" cancellations of the symbol, one-sided causality: evolution problem ill-posed backward in time, infinite regularisation forward in time, infinite-speed of propagation of information.

However some important equations of physics are transversal to this classification, e.g. the Schrödinger equation is parabolic-hyperbolic.