2. Vector spaces of functions

We consider vector spaces over the field $\mathbb{R}$ to simplify the presentation, most results carry on to $\mathbb{C}$ with minor modifications. We systematically illustrate this chapter with the Lebesgue spaces $L^p(\mathbb{R})$, $p \in [1, +\infty]$, the space of summable sequences $\ell^p(\mathbb{R})$, $p \in [1, +\infty]$, and spaces of continuous functions. (Note that $\ell^p(\mathbb{R})$ can be seen as the space of measurable functions $f : \mathbb{N} \to \mathbb{R}$ with ℓ^p summability, where the Borel sets in $\mathbb{N}$ are generated by the discrete topology, and standard theory of series follows from integration theory). We do not redo in classes the proofs of results in Linear Analysis but try to include proofs in the written notes. The decomposition on basis is postponed to the next chapter.

2.1. Recalls on normed vector spaces.

Definition 2.1 (Normed vector space)**.** *A* vector space *E (over $\mathbb{R}$) is a set that is closed under finite vector addition and scalar multiplication by a real. It is a* normed vector space *if furthermore there is a* norm *function $\|\cdot\| : E \to \mathbb{R}_+$ s.t. (1) $\|\alpha f\| = |\alpha|\|f\|$ for $\alpha \in \mathbb{R}$, $f \in E$ (homogeneity), (2) $\|f + g\| \leq \|f\| + \|g\|$ (triangular inequality), (3) $\|f\| = 0 \Rightarrow f = 0$ (w/o the latter it is a* semi-norm*).*

Example 2.2. *On a finite or countable set I, and for a sequence of positive weights $(w_i)_{i\in I}$: collections of reals with norm $\|x\|_{p,w} := (\sum_I |x_i|^p w_i^p)^{1/p}$ when $p \in [1, +\infty)$, or $\|x\|_{\infty,w} := \sup_I |x_i| w_i$ when $p = +\infty$. When $I = \mathbb{N}$, this is the space of p-summable sequence ℓ^p. Check the triangular inequality (Minkowski's inequality). More generally functions in $L^p(X)$ on a measured space X.*

Definition 2.3. *Then $(f, g) \mapsto \|f - g\|$ is a* distance*, inducing the* convergence*: $f_n \to f$ if $\|f_n - f\| \to 0$ as $n \to +\infty$. The sequence (f_n) of E is* Cauchy *if $\sup_{m,n\geq N} \|f_m - f_n\| \to 0$ as $N \to +\infty$.*

Exercise 17. *Every Cauchy sequence is bounded. Every convergent sequence is Cauchy. If (f_n) is a Cauchy sequence in E and there exists a subsequence $(f_{\varphi(n)})$ that converges to $f \in E$, then $f_n \to f$.*

Exercise 18. *Show that if $f_n \xrightarrow{\ell^p(\mathbb{R})} f$ in $\ell^p(\mathbb{R})$ implies that f_n converges to f* componentwise*: $\forall k \geq 1,\ f_n(k) \to f(k)$. Show that the converse is true if I finite but wrong in general if I is infinite.*

Definition 2.4 (Banach space)**.** *A normed vector space E is called a* Banach space *if it is complete, i.e., if every Cauchy sequence is convergent.*

Exercise 19. *We proved that $L^p(\mathbb{R})$ is a Banach space (Riesz-Fischer theorem). Prove directly (i.e. without using a generalised $L^p(X)$ version of the latter) that $\ell^p(\mathbb{R})$ is a Banach space.*

Exercise 20. *Prove that a vector normed space E is complete iff for any sequence (f_n), $\sum \|f_n\| < +\infty$ implies the convergence of the series $\sum f_n$.*

Definition 2.5 (Unconditional convergence)**.** *Let (f_n) be a sequence in E. The series $\sum f_n$ is said to* ***converge unconditionally*** *if every rearrangement of the series converges, i.e. for each bijection $\sigma : \mathbb{N} \to \mathbb{N}$ the series $\sum f_{\sigma(n)}$ converges.*

Exercise 21. *Prove that this definition is equivalent to: for every (ϵ_n) sequence in $\{-1, 1\}$ the series $\sum \epsilon_n f_n$ is convergent. Prove that if f_n converges unconditionally, then every rearrangement of the series must converge to the same sum. Prove that if E Banach space, the absolute convergence implies the unconditional convergence. Prove that the converse is true when E has finite dimension (Riemann rearrangement theorem) but fails in general (consider $f_n = e_n/n$ in ℓ^∞)*[2]*.*

Definition 2.6 (Subspace & closedness)**.** *A* ***subspace*** *F of the normed vector space E is a subset stable under vector addition and scalar multiplication. It is* closed *if $E \setminus F$ is an open set, i.e. for any $x \in E \setminus F$, there is some ball $B(x, r_x) \subset E \setminus F$.*

Exercise 22. *Prove that F subspace of E is closed iff it contains all its limit points. Prove that all subspaces are closed if E has finite dimension, but it is not true in general. [*Hint: Consider eventually vanishing sequences in $\ell^p(\mathbb{R})$.*]*

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[2] More generally the Dvoretzky-Rogers theorem asserts that every infinite-dimensional Banach space admits an unconditionally convergent series that is not absolutely convergent.

Example 2.7. *Prove that sequences converging to zero form a closed subspace of $\ell^\infty(\mathbb{R})$. Prove that continuous compactly supported functions (denoted $C^0_c(\mathbb{R})$) form a non-closed dense subspace of $L^p(\mathbb{R})$, $1 \le p < \infty$. Prove that continuous functions that goes to zero at infinity (denoted $C_0(\mathbb{R})$) form a closed subspace of $L^\infty(\mathbb{R})$, that is the closure of $C^0_c(\mathbb{R})$ in $L^\infty(\mathbb{R})$.*

Exercise 23. *A subspace F of a Banach space E is closed iff it is a Banach space.*

Definition 2.8 (Hilbert space). *A normed vector space E is said to be **pre-Hilbertian** if there is an **inner product** function $\langle \cdot, \cdot \rangle : E \to \mathbb{R}$ that is (1) bilinear, (2) s.t. $\langle x, x\rangle \ge 0$ and $\langle x, x \rangle = 0 \Rightarrow x = 0$. E is a **Hilbert space** if it is complete when endowed with the norm $\|x\| = \sqrt{\langle x, x\rangle}$.*

Exercise 24. *Prove that the inner product is uniquely determined by the norm it creates. Prove that for a norm $\|\cdot\|$ the existence of an inner product associated to it is equivalent to the parallelogram law: $\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2$ for any $f, g \in E$.*

Exercise 25. *Prove that if E is a normed vector space with finite dimension, all norms are equivalent (but not always Hilbertian) and induce a complete topology. Moreover the dimension of a normed vector space E is finite iff E is locally compact (i.e. the closed unit ball is compact). This crucial observation motivates the search for weaker topologies in infinite dimension in order to "restore Bolzano-Weierstrass".*

2.2. **Constructing enough continuous linear forms (Hahn-Banach theorem).** A natural step in understanding a functional space is to study how functions respecting the structure at play act on it, here the vectorial structure (linearity) and topology (continuity). The Hahn-Banach theorem shows that this *dual space* is not too small (in particular big enough to separate points, see later).

> **Theorem** (Hahn-Banach theorem) *Consider E normed vector space.*
> *(I) Given E_0 subspace of E and $F : E_0 \to \mathbb{R}$ linear continuous, there exists $\tilde F : E \to \mathbb{R}$ linear continuous extending F with $\|F\|_{E_0'} := \sup\{F(f) \mid f \in E_0,\ \|f\| \le 1\} = \|\tilde F\|_{E'} := \sup\{\tilde F(f) \mid f \in E,\ \|f\| \le 1\}$.*
> *(II) Let $A \subset E$ open convex not empty and $B \subset E$ convex not empty disjoint from A. There is a closed hyperplane weakly separating A and B, i.e. there is $F : E \to \mathbb{R}$ continuous and $\alpha \in \mathbb{R}$ s.t. $F \le \alpha$ on A and $F \ge \alpha$ on B.*
> *(III) Let $A \subset E$ closed convex not empty and $B \subset E$ compact convex not empty disjoint from A. There is a closed hyperplane strictly separating A and B, i.e. there is $F : E \to \mathbb{R}$ continuous and $\alpha \in \mathbb{R}$, $\varepsilon > 0$ s.t. $F \le \alpha - \varepsilon$ on A and $F \ge \alpha + \varepsilon$ on B.*

Remark 2.9. *When A and B are merely convex not empty disjoint, it is not always possible to separate them in infinite dimension (even weakly). It is however true in finite dimension (prove it).*

Proof (non-examinable). **(I) Algebraic form of HB.** We prove the slightly more general statement: *given $N : E \to \mathbb{R}$ a "subnorm" (i.e. $N(\lambda f) = \lambda N(f)$ for $\lambda > 0$ and $N(f+g) \le N(f) + N(g)$), a subspace $E_0 \subset E$ and $F : E_0 \to \mathbb{R}$ linear continuous s.t. $F \le N$ on G, then there is $\tilde F : E \to \mathbb{R}$ linear continuous extending F and s.t. $\tilde F \le N$ on E.* The statement implies (I) with $N(f) = \|F\|_{E_0'}\|f\|$.

The proof relies on *Zorn's lemma*: a partially ordered set containing upper bounds for every chain (that is, every totally ordered subset) necessarily contains at least one maximal element (this lemma is equivalent to the axiom of choice). Consider the set $\mathcal{P}$ of the pairs (E_H, H) with $G \subset E_H \subset E$ subspace, and $H : E_H \to \mathbb{R}$ linear continuous extending F and s.t. $H \le N$ on E_H. The set $\mathcal{P}$ is endowed with the partial order "extending the domain and the application". It is inductive (contains upper bounds for every chain) by taking the union of all domains and the application defined on each domain. It has a maximal element $(E_1, \tilde F)$. Let us prove that $E_1 = E$: if not there is $f_0 \in E \setminus E_1$ and we can define $\bar F(f + tf_0) = \tilde F(f) + t\alpha$ for $f \in E_1$ and $t \in \mathbb{R}$, and choose a constant α so that $\tilde F(f) + t\alpha \le N(f + tf_0)$. This choice is possible indeed: using that $N(\lambda f) = \lambda N(f)$ for $\lambda > 0$, and the vectorial structure of E_1, it is enough to prove

$$\left\{\forall f \in E_1, \quad \begin{matrix} F(f)+\alpha \le N(f+f_0) \\ F(f)-\alpha \le N(f-f_0) \end{matrix} \right\} \Leftrightarrow \forall f, g \in E_1, \; F(f) - N(f-f_0) \le \alpha \le N(g+f_0) - F(g)$$

and the latter is possible since $F(f) + F(g) = F(f+g) \le N(f+g) \le N(f-f_0) + N(g+f_0)$ from subtriangular inequality assumption on N. This negates then the maximality of $(E_1, \tilde{F})$, hence $E_1 = E$ and $\tilde{F}$ satisfies the requirements.

II. First geometric form of Hahn-Banach. First remark that for $F : E \to \mathbb{R}$ linear and any $\alpha \in \mathbb{R}$, the (affine) hyperplane $H = \{F = \alpha\}$ is closed iff F is continuous. The $\Leftarrow$ implication follows from continuity. To prove the $\Rightarrow$ implication: take f_0 s.t. $F(f_0) < \alpha$ and (H^c open) an open ball $B(f_0, \varepsilon)$ in H^c. Check that the ball is in $\{F < \alpha\}$ (else it would cross the hyperplane...) and finally $F(f_0 + \varepsilon g) \le \alpha$ for any $g \in B(0,1)$ hence $\|F\| \le \varepsilon^{-1}(\alpha - F(f_0))$.

Then the proof of (II) follows three steps: (1) for any C convex open in E containing 0 define its gauge $p(f) = \inf\{\alpha > 0 \mid \alpha^{-1} f \in C\} \in (0, +\infty)$. Check that it is a subnorm as above with $p(f) \le M\|f\|$ for some M uniform in $f \in E$, and $C = \{p < 1\}$; (2) for any convex open non-empty set $C \subset E$ and $f_0 \in E \setminus C$, one can separate C and $\{f_0\}$: take $F_0 : \mathbb{R} f_0 \to \mathbb{R}$ the linear form $F_0(tf_0) = t$ and $N = p$ the gauge of C and extend F_0 to F on E as in (I). Check that F is continuous and separates C and $\{f_0\}$; (3) for A and B as in (II) the set $C = A - B$ is convex open and does not contain $\{0\}$ as A and B disjoint, then separate C and $\{0\}$ by F from (2); finally check that F answers (II).

III. Second geometric form of Hahn-Banach. Check that there is $\varepsilon > 0$ small enough $A_\varepsilon := A + B(0, \varepsilon)$ and $B_\varepsilon = B + B(0,\varepsilon)$ (adding the open ball) are convex open disjoint non-empty, else there is $f_n \in A$, $g_n \in B$ s.t. $\|f_n - g_n\| \to 0$ and consider a converging a subsequence of $g_{\varphi(n)}$ in B, then $f_{\varphi(n)}$ converges as well the limit would be in $A \cap B$. Apply then (II) to separate A_ε and B_ε with some F, which separates strictly A and B. ☐

Remark 2.10. *The Hahn-Banach has many important applications: Krein-Milman theorem (a compact convex set K of a normed vector space E is the closure of the convex hull of its extremal points), theory of convex conjugated functions (see second example sheets), extension of continuous linear operators, existence of elementary solutions to linear constant coefficients PDEs...*

2.3. **Dual space and weak topologies.** Let us first review how basic topologies are generated.

Definition 2.11 (Initial topology)**.** *Consider a set X and $(Y_i)_{i \in I}$ topological spaces and $\varphi_i : X \to Y_i$. The* initial topology *generated by (Y_i, φ_i) on X is the weakest topology wrt which all φ_i are continuous.*

Exercise 26. *What is the initial topology generated a family of constant functions?*

The definition means: when ω_i describes the topology of Y_i and i describes I the sets $\varphi_i^{-1}(\omega_i)$ are open, and this is the coarsest ("most cost-effective in terms of open sets") topology that contains these open sets. Which leads us to the following problem: to construct the smallest family $\mathcal{T}$ of subsets of X that is stable by finite intersection and general unions and containing a given subfamily $\mathcal{F}$.

Exercise 27. *Check that such family exists. Check that it is obtained by (1) considering all finite intersections of elements of $\mathcal{F}$, (2) then all unions of elements of (1) [*Hint: The key point is to check that after these two operations the set of subsets constructed is stable by finite intersections.*] Check that reversing these two operations (first any unions, then finite intersections) does not work, and one would need to add a third step of "any union" again.*

Let us study neighbourhoods and convergence for the initial topology $\mathcal{T}$ generated on X.

Definition 2.12. *A* neighbourhood *V of $x \in X$ is a subset of X that contains an open set U that contains x. The* neighbourhood system *$\mathcal{V}(x)$ is the collection of all such neighbourhoods, and a* neighbourhood basis *$\mathcal{B}(x)$ is a subcollection s.t. any $V \in \mathcal{V}(x)$ contains a $B \in \mathcal{B}(x)$.*

Proposition 2.13. *Consider X and $(Y_i)_{i\in I}$ and $\varphi_i : X \to Y_i$ and the initial topology $\mathcal{T}$ on X as above, and any $x \in X$. Then a neighbourhood basis is given by*

$$\mathcal{B}(x) = \Big\{ \bigcap_{\text{finite}} \varphi_i^{-1}(\omega_i) \ \ \text{where } \omega_i \text{ open neighbourhood of } \varphi_i(x) \text{ in } Y_i \Big\}.$$

Proof. Let us first prove that the topology $\mathcal{T}$ on X is generated by the two operations in the previous exercise. Consider $\tilde{\mathcal{T}}$ the set of subsets of the form $\cup_{g\in G} \cap_{n=1}^{N_g} \varphi_{i_n}^{-1}(\omega_{i_n})$, where G can be identified wlog with $\{0,1\}^X$ by adding if necessary empty sets in the union. Then a finite union of such objects writes $\cap_{k=1}^{K}(\cup_{g\in G} A_{k,g})$ with $A_{k,g} := \cap_{n=1}^{N_{g,k}} \varphi_{i_{n,k}}^{-1}(\omega_{i_{n,k}})$ or empty, and rewrites $\cup_{g_1,\dots,g_K\in G} \cap_{k=1}^{K} A_{k,g_k}$ which is still an open set if non-empty. Hence $\tilde{\mathcal{T}}$ is stable by finite intersections. The stability by general union is clear from the two steps done (union done last) and thus $\tilde{\mathcal{T}}$ is a topology. Finally $\tilde{\mathcal{T}} \subset \mathcal{T}$ by since $\mathcal{T}$ is stable under the two operations performed and thus (minimality) $\tilde{\mathcal{T}} = \mathcal{T}$.

Back to the proof of the main statement: the given collection is made of open sets containing x so of neighbourhoods. Conversely a neighbourhood $V \in \mathcal{V}(x)$ contains an open set around x: $x \in U = \cup_{g\in G} \cap_{n=1}^{N_g} \varphi_{i_n}^{-1}(\omega_{g,i_n})$ with ω_{g,i_n} open in Y_{i_n} from above. Hence there is g_0 s.t. $x \in \cap_{n=1}^{N_{g_0}} \varphi_{i_n}^{-1}(\omega_{g_0,i_n})$ which concludes the proof. □

Exercise 28. *Deduce that $x_n \to x$ in X (for the topology $\mathcal{T}$ described above) iff $\varphi_i(x_n) \to \varphi_i(x)$ as $n \to \infty$, for all $i \in I$. Deduce also that if Z is another topological space and $\psi : Z \to X$, the application ψ is continuous iff $\varphi_i \circ \psi : Z \to Y_i$ continuous for all $i \in I$.*

Remark 2.14. *The* product topology *on $\otimes_{i\in I} X_i$ is obtained following the process above with the collection of spaces $(Y_i)_{i\in I}$, $Y_i = X_i$ (endowed with the topology of X_i) and applications $\varphi_j(x) = x_j$ for $x = (x_i)_{i\in I}$ from $\in \otimes_{i\in I} X_i$ to Y_j (canonical projections), and describe a neighbourhood basis and the convergence. Check that the* induced topology *on a subset is the initial topology for the canonical embedding. Quotient topologies are obtained as* final topologies *(finest s.t. $\varphi_i : Y_i \to X$ are cts...).*

Definition 2.15. *Consider E a normed vector space. The dual space E' is the space of linear continuous forms on E, i.e. $F : E \to \mathbb{R}$ linear continuous. The* weak topology $\sigma(E,E')$ *on E is the coarsest topology wrt the family of applications E' (vs the* strong topology *created by the norm).*

Exercise 29. *Describe a neighbourhood basis and the convergence for $\sigma(E,E')$.*

Proposition 2.16. *The topology $\sigma(E,E')$ on E is Hausdorff (distinct points can be separated by disjoint open neighbourhoods).*

Proof. Consider $f \neq g$ in E and apply the Hahn-Banach theorem with $A = \{f\}$ closed convex non-empty and $B = \{g\}$ compact convex non-empty: there is $\alpha \in \mathbb{R}$ and $F \in E'$ s.t. $f \in U_1 := \{F < \alpha\}$ open set and $g \in U_2 := \{F > \alpha\}$ open set. □

Exercise 30. *Prove that $\sigma(E,E')$ cannot be more refined than the strong topology on E. Prove that it agrees with the strong topology in finite dimension, but that in infinite dimension it is* always *strictly coarser (prove that weak closure of the unit sphere is the unit ball and/or that $B(0,1)$ is not weakly open and/or that the weak topology cannot be induced by a metric). By studying $\ell^1(\mathbb{R})$ show that it is nevertheless possible for some Banach spaces that weak and strong convergences agree.*

Definition 2.17. *E' is a normed vector space for the norm $\|F\|_{E'} = \sup\{|F(x)| \mid \|x\|_E \leq 1\}$. The weak-* topology $\sigma(E',E)$ on E' is generated by the applications $\varphi_f : E' \to \mathbb{R}$, $\varphi_f(F) = F(f)$, $f \in E$.*

Remark 2.18. *Observe that a simple consequence of Hahn-Banach (algebraic form (I)) is that $\|f\| = \max\{|F(f)| \mid F \in E',\ \|F\|_{E'} \leq 1\}$. Indeed RHS less than LHS by definition of the dual norm and by extending $F : \mathbb{R}f \to \mathbb{R}$, $F(tf) = t\|f\|^2$ one gets an $\tilde{F} \in E'$ s.t. $\tilde{F}(f) = \|f\|^2$ and $\|\tilde{F}\|_{E'} \leq \|f\|$ which saturates the inequality.*

Exercise 31. *Prove that $(E', \|\cdot\|_{E'})$ is complete (whether or not E is complete). Prove that $\sigma(E',E)$ cannot be finer than the strong topology on E'. Describe a neighbourhood basis and the convergence.*

Proposition 2.19. *The topology $\sigma(E', E)$ on E' is Hausdorff.*

Proof. No need to use the Hahn-Banach theorem here: consider $F_1 \neq F_2$ in E', there is $f \in E$ s.t. $F_1(f) \neq F_2(f)$, say wlog $F_1(f) < \alpha < F_2(f)$ with $\alpha \in \mathbb{R}$, then consider the open sets $U_1 := \varphi_f^{-1}((-\infty, \alpha))$ and $U_2 := \varphi_f^{-1}((\alpha, +\infty))$ to separate F_1 and F_2. □

Observe that we have now three topologies on E': denoting E'' the dual space of E', we can define the weak topology $\sigma(E', E'')$, the weak-* topolgy $\sigma(E', E)$ and the strong topology of the norm.

Exercise 32. *Prove that $\sigma(E', E)$ is coarser than $\sigma(E', E'')$ (show that the application $\Phi : E \to E''$, $f \mapsto \varphi_f$ maps the applications generating $\sigma(E', E)$ into a subset of those generating $\sigma(E', E'')$).*

Exercise 33. *Prove that if $\varphi : E' \to \mathbb{R}$ is continuous for $\sigma(E', E)$ then there is $f \in E$ s.t. $\varphi = \varphi_f$. [*Hint: find a neighbourhood V of zero in E' s.t. $|\varphi(V)| < 1$, where V in the neighbourhood basis.*]*

Theorem (Banach-Alaoglu-Bourbaki) *The closed unit ball of E' is weak-* compact ($\sigma(E', E)$).*

Exercise 34. *Prove this theorem starting from the Tychonoff theorem (any product of compact topological spaces is compact).*

Remark 2.20. *Prove-observe that for the topology of the norm the compactness of the closed unit ball is equivalent to the dimension being finite. Hence the importance of weaker topologies in infinite dimension and of the BAB theorem. In short, the fewer open sets the more compact sets (think to the definition of compactness by finite covering), but at the same time the fewer open sets the less precise the convergence is. General form of Tychonoff's theorem requires the axiom of choice. Prove however that a countable product of sequentially compact spaces is compact* without *the axiom of choice. As we shall see the axiom of choice is rarely needed to obtain compactness in* concrete *function spaces.*

2.4. **Reflexivity.** BAB theorem suggests to (try to) identify the E and E'' to get weak compactness.

Definition 2.21. *The Banach space E is said* reflexive *if $\Phi : E \to E''$, $f \mapsto \varphi_f$ is surjective.*

Exercise 35. *Prove that Φ is an isometry (using Hahn-Banach), when E'' is endowed with $\|\varphi\|_{E''} = \sup\{|\varphi(F)| \mid \|F\|_{E'} \leq 1\}$. Prove that finite-dimensional spaces are reflexive.*

Let us characterise reflexive spaces in terms of weak compactness:

Theorem 2.22 (Kakutani)**.** *A Banach space E is reflexive iff its closed unit ball is weakly compact.*

Proof. Direct implication (easier): the closed unit ball B_E is the pre-image $\Phi^{-1}(B_{E''})$ of that of E'', so it is enough to prove that Φ^{-1} is continuous wrt the topologies $\sigma(E'', E')$ and $\sigma(E, E')$. From exercise 28 the latter is equivalent to $F \circ \Phi^{-1}$ continuous (from $(E'', \sigma(E'', E'))$ to $\mathbb{R}$) for any $F \in E'$. Observe that for $\varphi \in E''$, $F \circ \Phi^{-1}(\varphi) = F \circ \Phi^{-1}(\varphi_f) = F(f) = \varphi_f(F) = \varphi(F)$. And given $F \in E'$, the application $\Psi_F : \varphi \in E'' \mapsto \varphi(F)$ is exactly one of the the applications generating $\sigma(E'', E')$, hence is continuous for this topology.

The reverse implication relies on a lemma due to Goldstine: *consider E Banach space then $\Phi(B_E)$ is dense in $B_{E''}$ for the weak-* topology $\sigma(E'', E')$.*

With this lemma at hand: Observe that Φ is continuous in strong topologies and thus for the weak topologies $\sigma(E, E')$ to $\sigma(E'', E''')$. Indeed this is equivalent to $\zeta \circ \Phi : (E, \sigma(E, E')) \to \mathbb{R}$ continuous for any $\zeta \in E'''$. But $\zeta \circ \Phi$ continuous for the strong topology, hence belongs to E' hence is weakly continuous by definition of $\sigma(E, E')$. As a consequence (weak-* being coarser than weak) Φ is continuous from $\sigma(E, E')$ to $\sigma(E'', E')$. Hence $\Phi(B_E) \subset B_{E''}$ is compact and dense in $\sigma(E'', E')$, hence $\Phi(B_E) = B_{E''}$ and thus $\Phi(E) = E''$.

Let us now prove the lemma: consider $\varphi \in B_{E''}$ and V neighbourhood of φ for $\sigma(E'', E')$, let us prove $\Phi(B_E) \cap V \neq \emptyset$. Write wlog (description of the topology) $V = \{\psi \in E''$ s.t. $\forall i =$

$1, \dots, n,\ |\psi(F_i) - \varphi(F_i)| < \varepsilon\}$ for $\varepsilon > 0$ and $F_1, \dots, F_n \in E'$. We search for $f \in E$ s.t. $|F_i(f) - \varphi(F_i)| < \varepsilon$ for $i = 1, \dots, n$. Denote $\alpha_i := \varphi(F_i)$ and observe that

$$\forall\, \beta_1, \dots, \beta_n \in \mathbb{R}, \quad \left|\sum \beta_i \alpha_i\right| = \left|\varphi\left(\sum \beta_i F_i\right)\right| \le \left\|\sum \beta_i F_i\right\|_{E'}. \tag{2.1}$$

Claim: It implies that there is $f \in B_E$ s.t. $|F_i(f) - \alpha_i| < \varepsilon$ for all $i = 1, \dots, n$, which concludes the proof of the lemma by picking $\varphi = \Phi(f) = \varphi_f$. Suppose the claim is wrong: consider $H : E \to \mathbb{R}^n$, $f \mapsto (F_1(f), \dots, F_n(f))$, then $(\alpha_1, \dots, \alpha_n) \notin \overline{H(B_E)}$. It implies (separation in $\mathbb{R}^n$) the existence of $\vec{\beta} = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$ s.t. $H(f) \cdot \vec{\beta} < \gamma < \vec{\alpha} \cdot \vec{\beta}$, which contradicts (2.1). □

Remark 2.23. *We proved that $\Phi(B_E)$ is dense in $B_{E''}$ for $\sigma(E'', E')$. Observe that is also closed in E'' for the strong topology: consider $\Phi(f_n)$ convergent in $(E'', \|\cdot\|_{E''})$ and thus Cauchy, since Φ is an isometry f_n is Cauchy and converges in E Banach space, hence $\lim \Phi(f_n) = \Phi(\lim f_n) \in \Phi(B_E)$. Hence $\Phi(B_E)$ is* never *dense in $B_{E''}$ when E is not reflexive.*

The Kakutani theorem is a powerful tool for "propagating" reflexivity.

Corollary 2.24. *(I) A closed subspace of a reflexive Banach space is reflexive (i.e. as a Banach space for the induced norm). (II) A Banach space E is reflexive iff E' is reflexive.*

Proof. (I) Consider M closed subspace of E, then check that $\sigma(E, E') \cap M = \sigma(M, M')$, then $\overline{B}_M$ is strongly closed hence $\sigma(E, E')$-closed, since B_E is $\sigma(E, E')$-compact, and by restriction it is $\sigma(M, M')$-compact, so by Kakutani theorem M is reflexive.

(II) If E reflexive, observe that $\sigma(E', E) = \sigma(E', E'')$ (generated by the same family of application). BAB theorem shows $B_{E'}$ is $\sigma(E', E)$-compact, hence $\sigma(E', E'')$-compact, so by Kakutani theorem E' is reflexive. Conversely, if E' is reflexive, then E'' is reflexive, and E is isometric to the closed subspace $\Phi(E)$ of E'' hence is reflexive. □

Most (but not all) concrete function spaces satisfy the following important geometric property:

Definition 2.25. *A Banach space E is said* uniformly convex *is for any $\varepsilon > 0$ there is $\delta > 0$ s.t.*

$$\forall\, f, g \in B_E, \quad \|f - g\|_E > \varepsilon \implies \left\|\frac{f+g}{2}\right\|_E < 1 - \delta.$$

Remark 2.26. *This property is* not *stable by equivalent of norm; it is satisfied by the Euclidean norm in $\mathbb{R}^n$ and more generally by any Hilbert norm (exercise). We will see later than $L^p(\mathbb{R})$, $p \in (1, +\infty)$ is UC, but not $L^1(\mathbb{R})$, $L^\infty(\mathbb{R})$, $C([0,1])$.*

Theorem 2.27 (Milman-Pettis)**.** *All uniformly convex Banach spaces are reflexive.*

Remark 2.28. *The converse is false (prove it using equivalent norms)*[3].

Proof. Since $\Phi(B_E)$ is closed for the strong topology in E'' (see above) it is enough to prove that $\Phi(B_E)$ is dense in $B_{E''}$ for the strong topology. Consider $\varphi \in E''$ with $\|\varphi\|_{E''} = 1$ and $\varepsilon > 0$, choose $\delta > 0$ of the def of UC and pick $F \in B_{E'}$ s.t. $\varphi(F) > 1 - \delta/2$. The set $V = \{\psi \in E''$ s.t. $|\psi(F) - \varphi(F)| < \delta/2\}$ is $\sigma(E'', E')$-open and Goldstine's lemma implies the existence of $f \in B_E$ s.t. $\Phi(f) \in V$. If $\|\Phi(f) - \varphi\|_{E''} \le \varepsilon$ we are done, else $\varphi \in W := (\Phi(f) + \varepsilon\overline{B_{E''}})^c$ and W is $\sigma(E'', E')$-open. Hence $V \cap W$ non-empty and $\sigma(E'', E')$-open: there is $g \in B_E$ s.t. $\Phi(g) \in V \cap W$. Then $f, g \in V$ implies $|\Phi(f)(F) - \varphi(F)| < \delta/2$ and $|\Phi(g)(F) - \varphi(F)| < \delta/2$, which implies $2\varphi(F) \le F(f+g) + \delta$ and thus $\varphi(F) \le \|\frac{f+g}{2}\|_E + \frac{\delta}{2}$, hence $\|\frac{f+g}{2}\|_E \ge 1 - \delta$ and (UC) $\|f - g\|_E \le \varepsilon$ which contradicts $g \in W$. □

Proposition 2.29. *Consider E uniformly convex Banach space, then $f_n \to f$ (strong convergence) is equivalent to [weak convergence + convergence of the norm].*

[3] In fact there are even reflexive separable Banach spaces that are not isomorphic to any uniformly convex space.

Proof. The direct implication is immediate. Reverse implication: consider $f_n \rightharpoonup f$ and $\|f_n\| \to \|f\|$ with $f \neq 0$ (else the conclusion is immediate), take n large enough so that the norms are non-zero, and define $g_n := f_n/\|f_n\|$ and $g := f/\|f\|$. Hence $(g_n + g)/2 \rightharpoonup g$ in $\sigma(E, E')$ and thus (standard) $1 = \|g\| \le \liminf \|\frac{g_n+g}{2}\|$, hence $\|\frac{g_n+g}{2}\| \to 1$ and (uniform convexity) $\|g_n - g\| \to 0$, which implies $f_n \to f$ strongly. □

2.5. **Separability and "countable Banach-Alaoglu-Bourbaki theorem".**

Definition 2.30. *X topological space is* separable *if it contains a dense countable subset.*

Exercise 36. *For X separable metric space show any $Y \subset X$ is separable for the induced topology.*

Proposition 2.31. *Let E Banach space with $(E', \|\cdot\|_{E'})$ separable. Then $(E, \|\cdot\|_E)$ is separable.*

Proof. Let (F_n) a sequence dense in E', and pick $f_n \in B_E$ s.t. $F_n(f_n) \ge \frac{1}{2}\|F_n\|$ for each n. The vector space L spanned by (f_n) is separable (consider the countable set of linear combinations of elements of (f_n) with rational coefficients), let us show that it is dense in E. If $F \in E'$ is zero on L, then there is n_0 s.t. $\|F - F_{n_0}\|_{E'} \le \varepsilon$, and $\|F_{n_0}\|_{E'} \le 2F_{n_0}(f_{n_0}) = 2(F_{n_0} - F)(f_{n_0}) \le 2\varepsilon$, and thus $\|F_{n_0}\|_{E'} \le 2\varepsilon$ and $\|F\|_{E'} \le 3\varepsilon$, and $(\varepsilon \to 0)$, $F = 0$. This implies that L is dense in E': otherwise apply Hahn-Banach's theorem form (II) with $A = \overline{L}$ and $B = \{f_0\}$ with $f_0 \in E \setminus \overline{L}$, which gives $F \in E'$ and $\alpha \in \mathbb{R}$ s.t. $F(f) < \alpha < F(f_0)$ for any $f \in L$ which implies $F = 0$ on L and $F \neq 0$. □

Remark 2.32. *In practice for concrete function spaces, the separability of E and E' is obtained by constructive arguments.*

Exercise 37. *Prove that a Banach space E is reflexive+separable iff its dual E' is reflexive+separable.*

Proposition 2.33. *A Banach space E is separable iff $(B_{E'}, \sigma(E', E))$ is metrizable.*

Proof. Direct implication: consider f_n a sequence dense in B_E and define on $E' \times E'$: $\text{dist}(F, G) := \sum_{n\ge1} 2^{-n}|F(f_n) - G(f_n)|$. This defines a distance: partial sums are bounded by $\|F - G\|_{E'}$, triangular inequality follows by summation, and $\text{dist}(F, G) = 0$ implies $F = G$ on the dense set $\{f_n\}$ of B_E, hence $F = G$ on E. This distance metrizes $\sigma(E', E)$ on $B_{E'}$: consider any neighbourhood for the distance $\{G \in B_{E'} \mid \text{dist}(F, G) \le \varepsilon\}$, then it contains the $\sigma(E', E)$-neighbourhood $\{G \in B_{E'} \mid |G(f_i) - F(f_i)| \le \varepsilon/2, i = 1, \dots, n_0\}$ with n_0 s.t. $2^{-n_0+1} \le \varepsilon/2$. Reciprocally, given a $\sigma(E', E)$-neighbourhood $U := \{G \in B_{E'} \mid |G(g_i) - F(g_i)| \le \varepsilon, i = 1, \dots, k\}$, then (density) consider f_{n_i} s.t. $\|f_{n_i} - g_i\|_E \le \varepsilon/3$, $i = 1, \dots, k$, and the distance neighbourhood $V := \{G \in B_{E'} \mid \text{dist}(F, G) \le 2^{-max|n_i|}\varepsilon/3\}$. Then $V \subset U$ as $\text{dist}(F, G) \le 2^{-max|n_i|}\varepsilon/3$ implies $|F(f_{n_i}) - G(F_{n_i})| \le \varepsilon/3$, $i = 1, \dots, k$, and by triangular inequality then $|G(g_i) - F(g_i)| \le |G(g_i) - G(f_{n_i})| + |G(f_{n_i}) - F(f_{n_i})| + |F(f_{n_i}) - F(g_i)| \le \varepsilon$, $i = 1, \dots, k$.

Reverse implication: for any $n \ge 1$ the set $U_n = \{F \in B_{E'} \text{ s.t. } D(F, 0) < 1/n\}$ is open around zero, hence there is $V_n = \{F \in B_{E'} \text{ s.t. } |F(f)| < \varepsilon_n,\ f \in C_n\} \subset U_n$ with $\varepsilon_n > 0$ and $C_n \subset E$ finite. The set $D = \cup C_n$ is countable and $\cap V_n = \{0\}$. Since $F_{|D} = 0$ implies $F \in \cap V_n$ i.e. $F = 0$, it follows as above that D is dense in E. □

Exercise 38. *Prove that E Banach space with E' separable implies that B_E is metrizable for $\sigma(E, E')$.*

Final step towards concreteness is a "cheap countable" version of Banach-Alaoglu-Bourbaki.

Proposition 2.34 ("Countable Banach-Alaoglu-Bourbaki")**.** *(I) Consider E **separable** Banach space, then closed bounded subsets of E' are sequentially weakly-* compact (i.e. for $\sigma(E', E)$).*

*(II) Consider E **reflexive** Banach space, then closed bounded subsets of E are sequentially weakly compact (i.e. for $\sigma(E, E')$).*

Proof. (I) It is enough to prove that any sequence F_n in $\overline{B}_{E'}(0, 1)$ has a subsequence of convergent for $\sigma(E', E)$. Consider f_k sequence dense in E: For each k, the real sequence $F_n(f_k)$, $n \ge 1$ is bounded in $\mathbb{R}$ hence has a convergent subsequence. By Cantor diagonal argument, there is a subsequence $F_{\theta(n)}$ s.t. for all $n \ge 1$, $F_{\theta(n)}(f_k)$ is convergent. Let $f \in E$, then for any $\varepsilon > 0$, there is f_{k_0} s.t. $\|f - f_{k_0}\| \le \varepsilon/3$

and since $F_{\theta(n)}(f_{k_0})$ is Cauchy, there is N s.t. $|F_{\theta(n_1)}(f_{k_0}) - F_{\zeta(n_2)}(f_{k_0})| \le \varepsilon/3$ for $n_1, n_2 \ge N$. Hence $|F_{\theta(n_1)}(f) - F_{\theta(n_2)}(f)| \le \varepsilon$ for $n_1, n_2 \ge N$ and thus $F_{\theta(n)}(f)$ is Cauchy in $\mathbb{R}$ and thus convergent.

(II) It is enough to prove that any sequence f_n in $\overline{B}_E(0,1)$ has a subsequence convergent for $\sigma(E, E')$. Let L be the closure of the vector space spanned by $\{f_n\}$ in E. It is separable (linear combinations of the f_n's with rational coefficients is dense in it) and reflexive (for instance observe that B_L is weakly compact by restriction and use Kakutani's theorem), hence L' separable and $(\overline{B}_L(0,1), \sigma(L, L'))$ identifies with $(\overline{B}_{L''}(0,1), \sigma(L'', L))$ hence (I) sequentially compact, which concludes the proof. □

2.6. **Study of concrete function spaces.** We proved that $L^p(\mathbb{R})$ is separable when $p \in [1, +\infty)$.

2.6.1. *The case of $L^p(\mathbb{R})$ with $p \in (1, +\infty)$.*

Theorem 2.35 (Riesz representation)**.** *The Banach space $L^p(\mathbb{R})$ with $p \in (1, +\infty)$ is reflexive and its dual space identifies isometrically with $L^{p'}(\mathbb{R})$, $p' := p/(p-1)$: for all $F \in (L^p(\mathbb{R}))'$ there is a unique $g \in L^{p'}(\mathbb{R})$ s.t. $F(f) = \int_{\mathbb{R}} fg$ for any $f \in L^p(\mathbb{R})$.*

Proof. Step 1. (First) Clarkson inequality. For $p \in [2, +\infty)$ it holds (extends parallelogram law)

$$\forall\, x, y \in \mathbb{R}, \qquad \left|\frac{x+y}{2}\right|^p + \left|\frac{x-y}{2}\right|^p \le \frac{|x|^p + |y|^p}{2}.$$

Indeed $\theta^{p/2} + (1-\theta)^{p/2} \le \theta + (1-\theta) \le 1$ for any $\theta \in [0,1]$, hence $a^p + b^p \le (a^2 + b^2)^{p/2}$ for any $a, b \ge 0$, hence with $a := |x+y|/2$ and $b := |x-y|/2$:

$$\left|\frac{x+y}{2}\right|^p + \left|\frac{x-y}{2}\right|^p \le \left(\left|\frac{x+y}{2}\right|^2 + \left|\frac{x-y}{2}\right|^2\right)^{p/2} = \left(\frac{x^2+y^2}{2}\right)^{p/2} \le \frac{|x|^p + |y|^p}{2}$$

(the final inequality comes from the convexity of $z \mapsto z^{p/2}$ since $p \ge 2$).

Step 2. Reflexivity when $p \in [2, +\infty)$. From the previous step integrating gives:

$$\forall\, f, g \in L^p(\mathbb{R}), \qquad \left\|\frac{f+g}{2}\right\|^p_{L^p(\mathbb{R})} + \left\|\frac{f-g}{2}\right\|^p_{L^p(\mathbb{R})} \le \frac{\|f\|^p_{L^p(\mathbb{R})} + \|g\|^p_{L^p(\mathbb{R})}}{2}.$$

Consider $\varepsilon > 0$ and $f, g \in L^p(\mathbb{R})$ with $\|f\| \le 1$, $\|g\| \le 1$ and $\|f - g\| > \varepsilon$ then $\|(f+g)/2\|^p_{L^p(\mathbb{R})} < 1 - (\varepsilon/2)^p$ which shows UC with $\delta = 1 - (1 - (\varepsilon/2)^p)^{1/p}$. Milman-Pettis' theorem implies reflexivity.

Step 3. Reflexivity when $p \in (1, 2]$. Define $\mathcal{F} : u \in L^p \mapsto F_u \in (L^{p'})'$ where $p' = p/(p-1)$ and $F_u(f) = \int_{\mathbb{R}} uf$ for $f \in L^{p'}$. The application is linear (linearity of the integral) and isometric: by Hölder's inequality $|F_u(f)| \le \|u\|_{L^p} \|f\|_{L^{p'}}$ hence $\|F_u\|_{(L^{p'})'} \le \|u\|_{L^p}$ and $f_u = |u|^{p-2} u \in L^{p'}$ with $\|f_u\|_{L^{p'}} = \|u\|^{p-1}_{L^p}$ and $F_u(f_u) = \int_{\mathbb{R}} |u|^p = \|u\|_{L^p} \|f_u\|_{(L^{p'})'}$ hence $\|F_u\|_{(L^{p'})'} = \|u\|_{L^p}$. This application $\mathcal{F}$ identifies L^p with a subspace of $(L^{p'})'$. This subspace is closed because of the isometry and completeness. We have proved in step 2 that $L^{p'}$ is separable reflexive, hence its dual is, hence a closed subspace of the latter is. And finally $\mathcal{F}(L^p)$ is reflexive, which shows that L^p is.

Step 4. Riesz representation theorem. Define $\mathcal{G} : u \in L^{p'} \mapsto G_u \in (L^{p'})'$ where $p' = p/(p-1)$ and $G_u(f) = \int_{\mathbb{R}} uf$ for $f \in L^p$. We prove as before $\|G_u\|_{(L^{p'})'} = \|u\|_{L^{p'}}$. It remains to prove that $\mathcal{G}$ is surjective. The image is closed (isometry and completeness), let us prove that it is dense. We use here the reflexivity: consider any $\varphi \in (L^p)''$ that is zero on the whole $\mathcal{G}(L^p)$, reflexivity implies $\varphi = \varphi_h = \Phi(h)$ with some $h \in L^p$, hence $\varphi_h(G_u) = G_u(h) = \int_{\mathbb{R}} uh$. By choosing $u = |h|^{p-2} h \in L^{p'}$ we get $\varphi(G_u) = \int_{\mathbb{R}} |h|^p = 0$ hence $h = 0$ and finally $\varphi = 0$. This shows the density and concludes the proof. □

Exercise 39. *In fact $L^p(\mathbb{R})$ is UC also when $p \in (1, 2]$, this follows from the 2d Clarkson inequality*

$$\forall\, f, g \in L^p(\mathbb{R}), \qquad \left\|\frac{f+g}{2}\right\|^{p'}_{L^p(\mathbb{R})} + \left\|\frac{f-g}{2}\right\|^{p'}_{L^p(\mathbb{R})} \le \left(\frac{\|f\|^p_{L^p(\mathbb{R})} + \|g\|^p_{L^p(\mathbb{R})}}{2}\right)^{\frac{1}{p-1}}.$$

2.6.2. *The case of* $L^1(\mathbb{R})$.

Theorem 2.36 (Riesz representation)**.** *The dual space of* $L^1(\mathbb{R})$ *identifies isometrically with* $L^\infty(\mathbb{R})$*: for all* $F \in (L^1(\mathbb{R}))'$ *there is a unique* $g \in L^\infty(\mathbb{R})$ *s.t.* $F(f) = \int_{\mathbb{R}} fg$ *for any* $f \in L^1(\mathbb{R})$.

Proof. The fact that any $g \in L^\infty$ creates an element $F : f \mapsto \int_{\mathbb{R}} fg$ of $(L^1)'$ follows from Hölder's inequality. Moreover $\|g\|_{L^\infty} = \|F\|_{(L^1)'}$: Hölder's inequality implies $\|F\|_{(L^1)'} \le \|g\|_{L^\infty}$ first and second $|\int_{\mathbb{R}} fg| \le \|f\|_{L^1}\|F\|_{(L^1)'}$ by definition of the dual norm, and assume $\{|g| > C > \|F\|_{(L^1)'}\}$ is not nullset: there is A measurable with non-zero finite measure where $|g| > C$, and take $f = \mathrm{sgn}(g)\chi_A$:

$$C\mu(A) \le \int_A |g| \,\mathrm{d}x \le \|F\|_{(L^1)'} \int_A \mathrm{d}x = \|F\|_{(L^1)'}\mu(A) \quad \text{which is a contradiction.}$$

Conversely, given $F \in (L^1)'$, let us construct such a unique g. (The fact that g is unique follows from the isometry property.) Define $w(x) = 1/(1 + |x|)$ and $G(h) = F(wh)$ for $h \in L^2$. This is a linear form on L^2 and $|G(h)| \le \|F\|_{(L^1)'}\|wh\|_{L^1} \le \|F\|_{(L^1)'}\|w\|_{L^2}\|h\|_{L^2}$ hence is continuous. From the previous theorem there is a unique $v \in L^2$ s.t. $G(h) = \int_{\mathbb{R}} vh$. Define $g = v/w$ (well-defined and measurable): when $f \in wL^2$ then $F(f) = G(f/w) = \int_{\mathbb{R}} vf/w = \int_{\mathbb{R}} gf$. Following the same argument as above with $f = w\mathrm{sgn}(g)\chi_A$ shows that g is essentially bounded $\|g\|_{L^\infty} \le \|F\|_{(L^1)'}$: if $\{|g| > C > \|F\|_{(L^1)'}\}$ on some A measurable with non-zero finite measure, take $f = w\mathrm{sgn}(g)\chi_A$, then

$$C \int_A w \le \int_A |g|w \le \|F\|_{(L^1)'} \int_A w \quad \text{which is a contradiction.}$$

It implies $\|g\|_{L^\infty} = \|F\|_{(L^1)'}$. Finally $C^0_c \subset wL^2$ is dense in L^1 hence F is represented by g on L^1. □

Corollary 2.37. *The space* $L^1(\mathbb{R})$ *is not reflexive.*

Proof. Consider an approximation of the unit e.g. $f_n = |B(0, 1/n)|^{-1}\chi_{B(0,1/n)}$. Assume L^1 reflexive then Kakutani theorem implies that $\overline{B}_{L^1}(0, 1)$ is weakly compact, i.e. for $\sigma(L^1, L^\infty)$. Hence (metrisable bc separable) there is a subsequence $f_{\theta(n)}$ s.t. for any $g \in L^\infty$, $\int f_{\theta(n)}g \to \int fg$ for some $f \in L^1$. The convergence with $g \in C^0_c(\mathbb{R} \setminus \{0\})$ implies $f = 0$ almost everywhere since $\int f_{\theta(n)}g \to \int fg = 0$ for n large enough (disjoint supports). The convergence with $g = 1$ implies $\int f = 1$, which is absurd. □

2.6.3. *The case of* $L^\infty(\mathbb{R})$. We know: $L^\infty(\mathbb{R})$ is the dual of $L^1(\mathbb{R})$ and $L^1(\mathbb{R})$ is separable hence the unit ball of $L^\infty(\mathbb{R})$ is weak-* compact in $\sigma(L^\infty, L^1)$ and metrizable and thus sequentially compact ("countable BAB"). However $L^\infty(\mathbb{R})$ is **not** reflexive unless $L^1(\mathbb{R})$ would be as well. Its dual is thus strictly larger than $L^1(\mathbb{R})$. This means: there exist continuous linear forms on $L^\infty(\mathbb{R})$ that are not represented by $L^1(\mathbb{R})$ functions. Possible examples are: (1) consider $F_n(f) := |B(0, 1/n)|^{-1} \int_{B(0,1/n)} f$ for $f \in L^\infty(\mathbb{R})$, the sequence is in $\overline{B}_{(L^\infty(\mathbb{R}))'}(0, 1)$ hence sequentially weak-* compact, however limits of subsequence cannot be represented by an L^1 function, (2) alternatively use Hahn-Banach theorem to extend the linear form $F(f) = f(0)$ on $C^0_c(\mathbb{R})$ to a continuous form on $L^\infty(\mathbb{R})$ and check that this extension cannot be represented by an L^1 function.

Remark 2.38. *The dual of* $L^\infty(\mathbb{R})$ *identifies with* ***finitely*** *additive finite signed measures on* $\mathbb{R}$ *absolutely continuous wrt the Lebesgue measure.*

Let us finally recall that $L^\infty(\mathbb{R})$ is **not** separable: consider the uncountable family $\chi_{[-r,r]}$, $r > 0$ in the unit ball. For any $r \ne r' > 0$, the distance $\|\chi_r - \chi_{r'}\|_{L^\infty} = 1$.

2.6.4. *The particular case of* $L^2(\mathbb{R})$. In Hilbert spaces there is another more geometric approach.

Proposition (Projection on a closed convex set) *Consider* H *Hilbert space,* $C \subset H$ *closed non-empty convex,* $x \in H$. *Then there exists a* unique $P_C(x) \in C$ *which realizes the minimization problem* $\|P_C(x) - x\| = \min_{y \in C} \|y - x\|$. *Moreover* P_C *is* 1*-Lipschitz.*

Proof. Set $x = 0$ w.l.o.g. If $0 \in C$ then we are done. If not denote $I := \inf_{y \in C} \|y\|$, and consider a minimizing sequence $y_n \in C$, $\|y_n\| \to I \geq 0$. The sequence is Cauchy:

$$\|y_m - y_n\|^2 = 2\|y_m\|^2 + 2\|y_n\|^2 - \|y_m + y_n\|^2 = 2\|y_m\|^2 + 2\|y_n\|^2 - 4\left\|\frac{y_m + y_n}{2}\right\|^2$$

and since $(y_m + y_n)/2 \in C$ by convexity, we have $\left\|\frac{y_m+y_n}{2}\right\|^2 \geq I^2$ and

$$0 \leq \|y_m - y_n\|^2 = 2\|y_m\|^2 + 2\|y_n\|^2 - 4\left\|\frac{y_m + y_n}{2}\right\|^2 \leq 2\|y_m\|^2 + 2\|y_n\|^2 - 4I^2 \xrightarrow{m,n\to\infty} 0.$$

This show the convergence $y_n \to y_\infty$ by completeness, and $y_\infty \in C$ by closedness. Hence the minimizer exists and $I > 0$ since $0 \notin C$. To prove that the the minimizer is unique, suppose by contradiction there are two distinct minimizers $y^1_\infty \neq y^2_\infty$, $y^1_\infty, y^2_\infty \in C$. Consider the mid-point $y^3_\infty = (y^1_\infty + y^2_\infty)/2$ and write (by expanding) $I^2 \leq \left\|y^3_\infty\right\|^2 = \frac{I^2}{2} + \frac{1}{2}\left\langle y^1_\infty, y^2_\infty\right\rangle$ hence $I^2 \leq \left\langle y^1_\infty, y^2_\infty\right\rangle$. This implies since $I^2 = \|y^1_\infty\|^2 = \|y^2_\infty\|^2$ that $\left\langle y^1_\infty, y^2_\infty - y^1_\infty\right\rangle \geq 0$ and $\left\langle y^1_\infty - y^2_\infty, y^2_\infty\right\rangle \geq 0$ and thus $\left\|y^1_\infty - y^2_\infty\right\| = 0$.

Consider $t \in (0,1)$, two points $x_1, x_2 \in H$ and their projections $P_C(x_1), P_C(x_2) \in C$, and write

$$\begin{aligned}\|P_C(x_2) - x_2\|^2 &\leq \|tP_C(x_1) + (1-t)P_C(x_2) - x_2\|^2 \\ &= \|P_C(x_2) - x_2\|^2 + t^2\|P_C(x_2) - P_C(x_1)\|^2 + 2t\langle P_C(x_2) - x_2, P_C(x_1) - P_C(x_2)\rangle\end{aligned}$$

$$\begin{aligned}\|P_C(x_1) - x_1\|^2 &\leq \|tP_C(x_2) + (1-t)P_C(x_1) - x_1\|^2 \\ &= \|P_C(x_1) - x_1\|^2 + t^2\|P_C(x_2) - P_C(x_1)\|^2 + 2t\langle P_C(x_1) - x_1, P_C(x_2) - P_C(x_1)\rangle\end{aligned}$$

which use the fact that $tP_C(x_1) + (1-t)P_C(x_2) \in C$, $tP_C(x_2) + (1-t)P_C(x_1) \in C$. We deduce

$$t^2\|P_C(x_2) - P_C(x_1)\|^2 + 2t\langle P_C(x_2) - x_2, P_C(x_1) - P_C(x_2)\rangle \geq 0$$
$$t^2\|P_C(x_2) - P_C(x_1)\|^2 + 2t\langle P_C(x_1) - x_1, P_C(x_2) - P_C(x_1)\rangle \geq 0$$

and by summing the two inequalities

$$2t^2\|P_C(x_2) - P_C(x_1)\|^2 \geq 2t\langle P_C(x_2) - x_2 + x_1 - P_C(x_1), P_C(x_2) - P_C(x_1)\rangle$$

which implies $\|P_C(x_2) - P_C(x_1)\| \leq \frac{1}{1-t}\|x_2 - x_1\|$ and $(t \to 0)$ it shows the Lipschitz bound. □

Theorem (Riesz representation in Hilbert spaces) *Consider a Hilbert space H and a continuous linear form $F : H \to \mathbb{R}$. Then there is a unique $y \in H$ such that $F(x) = \langle x, y\rangle$ for all $x \in H$. Moreover the map $F \in H' \mapsto y \in H$ is linear and isometric.*

Proof. If $F = 0$ then $y = 0$ is the only solution and we are done. If F is non-zero, then $M = F^{-1}(\{0\})$ is a closed strict subspace of H, there is $x_1 \in H$, $\|x_1\|_H = 1$ and $x_1 \perp M$: take any $x_0 \notin M$, then $x = (P_M(x_0) - x_0)/\|P_M(x_0) - x_0\|$. Then $F = F(x_1)\langle x_1, \cdot\rangle$. □

2.6.5. *Spaces of continuous functions.* We have seen that the space of continuous compactly supported functions $C^0_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $p \in [1, +\infty)$. In $L^\infty(\mathbb{R})$, its closure is the set $C_0(\mathbb{R})$ of continuous functions that go to zero at infinity. To preserve continuity it is natural to use the supremum norm (uniform convergence), there are two naturally closed spaces of continuous functions: those vanishing at infinity C_0 and those merely bounded C_b. On a bounded closed interval they coincide. We have seen that the dual of $L^\infty(\mathbb{R})$ is more complicated than a space of functions (or even measures), however we can understand the duals of its subspaces of continuous functions.

Theorem (Riesz-Markov) *The dual of $C_0(\mathbb{R})$ is $\mathcal{M}(\mathbb{R})$ the space of real valued Radon measures (locally finite regular Borel measures) on $\mathbb{R}$. Namely the following map is an isometric isomorphism between this dual and $\mathcal{M}(\mathbb{R})$: $\mu \mapsto F_\mu(f) = \int f \, \mathrm{d}\mu$ and $\|F_\mu\| = \|\mu\|_{TV}$ the total variation of μ defined by $\sup_{j=1}^n |\mu(A_j)|$ supremum over all collection $A_1, \dots, A_m$ of disjoint subsets of $\mathbb{R}$.*

Remark 2.39. *The previous theorem applies similarly on an interval. When the linear form F is* positive *i.e. $F(f) \geq 0$ when $f \geq 0$, the measure that represents f is a regular Borel measure (Riesz-Markov-Kakutani); an extension of the representation to $C_b(\mathbb{R})$ then makes sense by restriction.*

2.7. **Baire's theorems and local-to-global arguments.** The Baire lemma is a powerful tool to measure how *small or large* a set is, hence obtaining existence or global bounds from local properties.

Lemma (Baire) *Consider X complete metric space. Then (i) a countable intersection $\cap O_n$ of dense open sets O_n is dense, (ii) a countable union $\cup C_n$ of closed sets C_n with empty interior has empty interior, (iii) if $X = \cup C_n$ (countable union) of closed subsets X_n, then at least one of the subsets has non-empty interior.*

Proof. The equivalence of (i) and (ii) follows from taking complement sets, and (iii) follows from (ii). To prove (i) build by induction a decreasing sequence of balls s.t. $\overline{B}(x_{n+1}, r_{n+1}) \subset B(x_n, r_n) \cap O_{n+1}$ with initialisation $B(x_0, r_0)$ any open ball, using the density and openness of the O_n's. Since such sequence x_n is Cauchy, its limit exists, is in $\cap O_n$ and in the initial ball, which proves the density. □

Exercise 40. *Prove that (i)-(ii)-(iii) still hold in locally compact Hausdorff topological spaces. Prove with (iii) that every complete metric space with no isolated points is uncountable. Prove with Baire's lemma the existence of continuous, nowhere differentiable functions (argument due to Banach).*

Remark 2.40. *When the space is separable Baire's lemma does not require the axiom of choice.*

A key example of local-to-global argument based on Baire's lemma is:

Theorem (Banach-Steinhaus) *Consider E_1 and E_2 Banach spaces:*
(I) Given T_i, $i \in I$ a family (non necessarily countable) of linear continuous applications from E_1 to E_2 s.t. $\forall f \in E_1$, $\sup_{i \in I} \|T_i f\|_{E_2} < \infty$, there exists $C > 0$ s.t. for all $f \in E_1$, $\sup_{i \in I} \|T_i f\|_{E_2} \leq C \|f\|_{E_1}$ (uniform boundedness).
(II) Given T_n a sequence of linear continuous applications from E_1 to E_2 s.t. for any $f \in E_1$, $T_n f$ converges to some limit Tf in E_2 (pointwise convergence), then T is linear continuous with $\sup_n \|T_n\| < \infty$ and $\|T\| \leq \liminf_n \|T_n\|$.
(III) A subset B of E_1 is bounded iff for all $F \in E_1'$, $F(B) \subset \mathbb{R}$ is bounded.
(IV) A subset B' of E_1' is bounded iff for all $f \in E_1$, $\varphi_f(B') \subset \mathbb{R}$ is bounded.

Proof. (I) Define $V_n = \{f \in E_1 \mid \forall i \in I, \ \|T_i f\|_{E_2} \leq n\}$: it is closed and $E_1 = \cup_n V_n$ from the assumption, hence $\exists n_0$ s.t. V_{n_0} has not empty interior, i.e. there is $B_{E_1}(f_0, r_0) \subset V_{n_0}$, which implies that $\|T_i f\|_{E_2} \leq r_0^{-1}(n_0 + \|T_i f_0\|)$ for any $f \in B_{E_1}(0,1)$ and $i \in I$, which concludes the proof using $\sup_I \|T_i f_0\| < \infty$. (II) Apply (I) to the sequence T_n. (III) The $\Rightarrow$ implication follows from linear continuity, the $\Leftarrow$ implication follows from (I) with $E_1 = E'$, $E_2 = \mathbb{R}$, $I = B$ and the family of application φ_f for $f \in B$. (IV) The $\Rightarrow$ implication follows from linear continuity, the $\Leftarrow$ implication follows from (I) with $E_1 = E$, $E_2 = \mathbb{R}$, $I = B'$ and the family of application F for $F \in B'$. □

Remark 2.41. *(III) means "weakly bounded" equivalent to "strongly bounded" or in other words: to check that a set is bounded it is enough to check along each linear continuous forms. This replaces* *the idea of checking each coordinate in finite dimension. Observe also that (III) shows that weak convergence implies boundedness, and (IV) implies that weak-* convergence implies boundedness.*

The other key example is the regularity of the inverse map for linear continuous applications.

Theorem (Banach) *(I) Given E_1 and E_2 two Banach spaces and $T : E_1 \to E_2$ linear continuous surjective (onto), then T is open, i.e. maps open sets to open sets.*
(II) Given E_1 and E_2 two Banach spaces and $T : E_1 \to E_2$ linear continuous bijective, then T^{-1} is continuous.
(III) Given E_1 and E_2 two Banach spaces and $T : E_1 \to E_2$ linear, then T continuous iff its ***graph*** *$G(T) := \{(f, Tf) \mid f \in E_1\} \subset E_1 \times E_2$ is closed.*

Proof. (I) implies (II): $T(B_E(0,1))$ contains some $B_F(0,c)$ which implies $\|f\|_{E_1} \le c^{-1}\|Tf\|_{E_2}$ for any $f \in E_1$ which shows that T^{-1} is continuous. Remark that it implies if two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are given on E normed vector space s.t. both make E complete (Banach space) and one is dominated $\|\cdot\|_1 \le C\|\cdot\|_2$ for some $C > 0$ then they are equivalent i.e. there is also $C' > 0$ s.t. $\|\cdot\|_2 \le C'\|\cdot\|_2$.

(II) implies (III): The $\Rightarrow$ implication follows from continuity. For the $\Leftarrow$ implication: The two norms $\|f\|_1 := \|f\|_{E_1} + \|Tf\|_{E_2}$ and $\|f\|_2 := \|f\|_{E_1}$ make E_1 a Banach space (for $\|\cdot\|_1$ use that the graph is closed to get completeness) and $\|\cdot\|_1$ dominates $\|\cdot\|_2$ hence $\|\cdot\|_2$ dominates $\|\cdot\|_1$ which shows that T is continuous.

Proof of (I): By linearity it is enough to prove that $T(B_{E_1}(0,1))$ contains some $B_{E_2}(0,c)$, $c > 0$. First step there is $c > 0$ s.t. $\overline{T(B_{E_1}(0,1))}$ contains some $B_{E_2}(0,2c)$: $\cup S_n = E_1$ with $S_n = n\overline{T(B_{E_1}(0,1))}$ hence there is n_0 s.t. S_{n_0} is non-empty interior which implies that $\overline{T(B_{E_1}(0,1))}$ has non-empty interior, hence contains some $B_{E_2}(g_0, 4c)$ which implies since $-g_0 \in T(\overline{B}_{E_1}(0,1))$ that $B_{E_2}(g_0, 2c) \subset \overline{T(B_{E_1}(0,1))}$. Second step for $g \in B_{E_2}(0,c)$ we construct $f \in B_{E_1}(0,1)$ s.t. $T(f) = g$: there is $f_0 \in B_{E_1}(0,1/2)$ s.t. $\|T(f_0) - g\|_{E_2} \le c/2$, then there is $f_1 \in B_{E_1}(0,1/4)$ s.t. $\|T(f_1) + T(f_0) - g\|_{E_2} \le c/4$, and by induction there is $f_{n+1} \in B_{E_1}(0, 1/2^n)$ s.t. $\|T(f_{n+1}) + \cdots + T(f_0) - g\|_{E_2} \le c/2^n$. The series $\sum f_n$ is absolutely convergent hence convergent (completeness) and its sum f satisfies $\|f\|_{E_1} < 1$ and $T(f) = g$. □

Baire theory vs. measure theory to assess smallness. The Baire theory is often formulated in terms of the following "topological sizes" of sets: consider X a topological space, a subset B of X is **nowhere dense** if there is no neighbourhood on which B is dense: for any nonempty open set U in X, there is a nonempty open set V contained in U such that V and B are disjoint. The complement of a nowhere dense is a set with dense interior, or in other words, a nowhere dense set is "full of holes" . A subset A of X is **meagre** if it can be expressed as the union of countably many nowhere dense subsets; a **comeagre** set is one whose complement is meagre, or equivalently, the intersection of countably many sets with dense interiors[4]. Baire's lemma says that a meager space cannot fill up the whole space (assuming the whole space to be complete metric).

Exercise 41. *Prove that $\mathbb{Q}$ is meagre in $\mathbb{R}$, but* not *nowhere dense. Prove that the set of functions which have a derivative at some point is a meagre set in the space of all continuous functions.*

In measure theory, the natural notion of small set is that of **nullset**, i.e. in $\mathbb{R}$ it is a set that can be covered by a countable union of intervals of arbitrarily small total length. These two notions (meager and nullset) are "orthogonal", in the sense that one can be small for measure and not for Baire and vice-versa. As an example let us prove that $\mathbb{R}$ can be partitioned into two subsets, one a nullset and the other one meager: for each $n \ge 1$ let N_n a union of disjoint open intervals that covers $\mathbb{Q}$ and whose total length does not exceed 2^{-n}. Then $N = \cap_n N_n$ is a nullset, but at the same time it is an intersection of open dense sets, thus comeager, hence its complement is meager.

[4]A meagre set is also called a set of first category; a nonmeagre set is also called a set of second category (second category does *not* mean comeagre).

2.8. **Compactness.** In the hunting for compactness we have now:

(1) Bolzano-Weierstrass in $(\overline{B}_E(0,1), \|\cdot\|)$ in finite dimension,
(2) "Countable BAB" in separable spaces: $(\overline{B}_{E'}(0,1), \sigma(E',E))$ sequentially compact,
(3) "Countable BAB" in reflexive spaces: $(\overline{B}_E(0,1), \sigma(E,E'))$ sequentially compact.

This gives sequential weak and weak-* compactness of closed unit balls of $L^p(\mathbb{R})$, $p \in (1,+\infty)$, and sequential weak-* compactness of the unit balls of $L^\infty(\mathbb{R})$. We miss two important things: (1) criteria for strong compactness (Arzelà-Ascoli), (2) weak compactness in $L^1(\mathbb{R})$ (Dunford-Pettis).

Exercise 42. *Consider f_n a sequence bounded in $L^p(I)$ with $p \in (1,+\infty]$ and I bounded open interval, and s.t. $f_n \to f$ almost everywhere. Prove that $f_n \to f$ strongly in $L^q(I)$ for any $q \in [1,p)$.*

Theorem 2.42 (Arzelà-Ascoli)**.** *A sequence f_n of continuous functions on a compact metric space X is relatively compact in the topology induced by the uniform norm iff it is (1) **equicontinuous**: for any $\varepsilon > 0$ and $x \in X$, there is V neighborhood of x so that $\forall\, n \geq 1$, $\forall\, y \in V$, $\quad |f_n(x) - f_n(y)| \leq \varepsilon$, and (2) **equibounded (pointwise)**: for any $x \in X$, $\sup_{n\geq 0} |f_n(x)| < +\infty$.*

Remark 2.43. *The boundedness assumption is reminiscent of weak compactness statements. However the equicontinuity is an assumption of uniform regularity along the sequence that proscribes the possibility of oscillatory behaviors as n goes to infinity. As a general principle, the main obstacle to weak compactness is the divergence, and the main obstacle to a weak compactness being strong is* oscillations *(i.e. divergence in Fourier variable) which can be ruled out by uniform regularity assumption.*

Proof. Construct a countable dense subset Y of X by using the compactness of X and finite coverings extracted from small balls around every points, then show convergence of a subsequence of $f_n(x)$ for each $x \in Y$ by Bolzano-Weierstrass, and find a subsequence converging for all points of Y by a diagonal argument. This defines a limit f on Y. The equicontinuity implies then the continuity of this limit on Y. It can therefore be extended by density to a continuous function on X. The uniform convergence finally follows from the equicontinuity and the pointwise convergence. □

Exercise 43. *Give example of sequences converging weakly but not strongly in L^p. Show that the weak limit of a product is not in general the product of the weak limits.*

Theorem 2.44 (Riesz-Fréchet-Kolmogorov)**.** *Consider a sequence f_n bounded in $L^p(\mathbb{R})$, $p \in [1,+\infty)$ s.t. for any $\varepsilon > 0$ there is $\delta > 0$ s.t. $\sup_n \|\tau_h f_n - f_n\|_{L^p([0,1])} < \varepsilon$ for $|h| < \delta$. Then it has a subsequence converging strongly in $L^p([0,1])$.*

Proof. Consider an approximation of the unit $\varphi_k(x) = k\varphi(x/k)$ with $\varphi \geq 0$ even, $\operatorname{supp}\varphi = (-1,1)$ and $\int_{\mathbb{R}} \varphi = 1$. With (ε,δ) as in the assumptions, $\sup_n \|\varphi_k * f_n - f_n\|_{L^p([0,1])} < \varepsilon$ when $k > 1/\delta$. For each given $k \geq 1$, the sequence $(\varphi_k * f_n)_{n\geq 1}$ satisfies the assumptions of the Arzelà-Ascoli theorem on $[0,1]$: equibounded by Hölder's inequality, and equicontinuous by $|\varphi_k * f_n(x_1) - \varphi_k * f_n(x_2)| \leq \|\varphi_k(\cdot - x_1) - \varphi(\cdot - x_2)\|_{L^{p'}} \|f_n\|_{L^p} \leq |x_1 - x_2| \|\varphi_k'\|_{L^{p'}} \|f_n\|_{L^p}$. Hence (Cantor diagonal argument), there is a subsequence $f_{\theta(n)}$ s.t. $\varphi_k * f_n$ converges uniformly and thus $L^p([0,1])$ for any k. Together with the above convergence in k, it shows that $f_{\theta(n)}$ is Cauchy $L^p([0,1])$ and concludes the proof. □

Theorem 2.45 (Dunford-Pettis)**.** *A sequence of functions f_n in $L^1(\mathbb{R})$ is weakly compact iff it is (1) **bounded** in $L^1(\mathbb{R})$, (2) **tight**: for any $\varepsilon > 0$ there is $M > 0$ s.t. $\int_{\mathbb{R}\setminus[-M,M]} |f_n| < \varepsilon$ for all n, (3) **uniformly integrable**: for any $\varepsilon > 0$ there is $\delta > 0$ s.t. for any measurable set A with $\mu(A) < \delta$, $\int_A |f_n| < \varepsilon$ for all n.*

Proof. Non-examinable (see example sheet). □

Exercise 44. *Prove that the two assumptions on f_n are equivalent to the existence of $\omega : \mathbb{R} \to (0,+\infty)$ continuous and going to infinity at infinite, and $\mathcal{N} : \mathbb{R}_+ \to \mathbb{R}_+$ measurable with $\mathcal{N}(s)/s \to +\infty$ as $s \to +\infty$, s.t. $\sup_n \int_{\mathbb{R}} (\omega \cdot |f| + \mathcal{N}(|f|)) < \infty$.*