KINETIC THEORY

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1. The birth of kinetic theory

Modern physics can be traced back to Newton, with the advent of differential equations to substantiate the laws of classical mechanics. In the following centuries this was followed by more comprehensive theories of physical phenomena in the surrounding world: electric and magnetic forces were captured by the theory of electromagnetism (Ampère, Faraday, Maxwell); large velocities were handled by the theory of relativity (Lorentz, Poincaré, Minkowski, Einstein); small-scale particle physics offered itself to quantum mechanics (Planck, Einstein, Bohr, Heisenberg, Born, Jordan, Pauli, Fermi, Schrödinger, Dirac, De Broglie, Bose), etc.

However, all these theories are classically devised to study one physical system (planet, ship, motor, battery, electron, spaceship, etc.) or a small number of systems (planets in the Solar system, electrons in a molecule, etc.) In many situations though, one needs to deal with an assembly made up of elements so numerous that their individual tracking is neither useful nor possible: galaxies made of hundreds of billions of stars, fluids made of more than 10^{20} molecules, crowds made of thousands of individuals, etc. Taking such large numbers into account leads to new effective laws of physics, requiring different models and concepts. This passage from microscopic rules to macroscopic laws is the founding principle of statistical physics. All branches of physics (classical, quantum, relativistic, etc.) can be studied from the point of view of statistical physics, in both stationary and dynamical perspectives. Classical mechanics was naturally one of the first laboratories for statistical physics, and thus in the nineteenth century was born kinetic theory.

Before we describe the key concepts of kinetic theory, let us recall the basic notion of *phase space*, which should be thought of as the space of all possible states occurring in a mathematical model of some physical system. If one studies a deterministic system obeying an evolution equation, then the phase space is, in principle, the "smallest" space on which the equation determines a unique, "well-behaved" solution. For instance, the evolution of a classical point particle is governed by a second-order differential equation (Newton's law); so the position of the particle is not sufficient to predict its future positions, but the pair (position, velocity) is sufficient to predict future positions and velocities (let alone particular situations such as initial conditions leading to collisions between point particles, etc.). Thus the phase space of a classical particle is made of positions and velocities. On the other hand, if the physical system is, for instance, a rigid body with a certain shape, then the phase space should also include extra parameters related to the orientation of the body.

The main idea in kinetic theory is to replace a huge number of objects, whose physical states are completely described by points in a certain phase space, and whose properties are otherwise identical, by a *statistical distribution* over that phase space. In particular, a large crowd of classical point particles will be described by a statistical distribution on the space of positions and velocities.

In retrospect, the conceptual leap from Newtonian mechanics to kinetic theory was quite significant: the new formalism involved a set of invisible variables, namely the velocities of particles, which are inaccessible to observation. It was even counterintuitive: for instance, kinetic theory replaces the model of a fluid at rest (zero velocity) by a huge number of particles moving in all directions with great speed. This increase in complexity was not easy to justify, since at the time there was no way to measure any of these velocities — it is still hardly possible today. This fundamental role of velocities accounts for the denomination of *kinetic* theory.

With kinetic theory came the distinction between three scales: the macroscopic scale of phenomena which are accessible to observation; the microscopic scale of molecules and infinitesimal constituents; and an intermediate scale, loosely defined and often called *mesoscopic*. This is the scale of phenomena which are not accessible to macroscopic observation but already involve a large number of particles, so that statistical effects are significant.

With a little stretch, one can compare the principles of kinetic theory to certain contemporary models of theoretical physics, such as string theory, in which a set of hypothetical hidden variables is also taken into account (this is putting aside any debates about the value and possibility of validation of string theory).

The basic scheme of kinetic theory leaves room for obvious variations. If there are several species, one can consider several statistical distributions. (Think of air, which is mainly made of a mixture of two gases; the two species have different properties, but within each species the molecules can be considered as identical.) If the position and velocity are not sufficient to describe the state of one object, one can enlarge the phase space. (In the case of air, one might wish to keep track of the orientation of a molecule of nitrogen or oxygen gas.)

Kinetic theory was first discussed by D. Bernoulli in the 18th century. The notions of *mean free path* and *mean free time*, which are the typical distance and typical time, respectively, that a particle can travel without hitting another one, were studied by various authors (Herapath, Waterston, Joule, König, Clausius) from 1820 to 1860. At the same time emerged the most important notion of *cross section*, which measures the likelihood of interaction between two particles, and can be interpreted as an effective collision surface. However, the field as we know it was really founded by Maxwell in a celebrated 1867 paper.

This theory was strongly influenced by two major earlier scientific developments. The first one was the rise of thermodynamics, all throughout the 18th and 19th century. The laws governing exchanges of energy and variations of heat, density, pressure and temperature, did not seem to rely on fundamental equations and were discovered through a slow and confusing process; thus it was desirable to grasp some more fundamental laws that would underlie thermodynamics. The second influence was the development of statistics, especially in the field of social sciences, with the empirical discovery by Galton and Quetelet of the omnipresence of simple statistical laws derived from probability theory — one main example was the recognition that fluctuations of the size of individuals was essentially governed by Gaussian distributions. Both of these influences were already guiding Bernoulli (whose father was one of the founders of probability theory), but by the time of Maxwell they had become much more mature.

In those days the atomistic nature of matter was still largely hypothetical, and kinetic theory could be considered as a thought experiment. Maxwell discussed the problem of the derivation of macroscopic laws from microscopic physics. He worked in a dilute regime to neglect collisions involving more than two bodies, and assumed a clear separation between the inhomogeneity scale (a mesoscopic concept) and the interaction scale (at microscopic level). He then computed the effect of collisions on the distribution function via the solution of a classical scattering problem. In this way he came up with an evolution equation, equivalent to what we now call the **Boltzmann equation**. The unknown is a density function $f(t, x, v)$, standing for the density of particles at time t in the phase space (x, v) (equipped with the reference Liouville measure $dx\,dv$); and the equation, in modern writing, is

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F(t,x) \cdot \nabla_v f = Q(f,f). \tag{1}$$

Here the left-hand side describes the evolution of f under the action of the force $F(t, x)$, while the action of elastic collisions is described by the nonlinear operator Q on the right-hand side:

$$Q(f,f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \widetilde{B}(v - v_*, \omega) \Big(f(t,x,v')\, f(t,x,v'_*) - f(t,x,v)\, f(t,x,v_*) \Big)\, dv_*\, d\omega. \tag{2}$$

Note that this operator is localized in t and x, quadratic, and has the structure of a tensor product with respect to $f(t, x, \cdot)$. The velocities v' and v'_* should be thought of as the velocities of a pair of particles before collision, while v and v_* are the velocities after that collision: the formulas are

$$v' = v - \langle v - v_*, \omega\rangle\, \omega, \qquad v'_* = v_* + \langle v - v_*, \omega\rangle\, \omega.$$

When one computes (v, v_*) from (v', v'_*) (or the reverse), conservation laws are not enough to yield the result, with only 4 scalar conservation laws for 6 degrees of freedom. The unit vector $\omega \in \mathbb{S}^2$ removes this ambiguity: in the case of colliding hard spheres, it can be thought of as the direction of the line joining the two centers of the particles. The kernel $\widetilde{B}(v - v_*, \omega)$ describes the relative frequency of vectors ω, depending on the relative impact velocity $v - v_*$; it only depends on the modulus $|v - v_*|$ and the deflection angle θ between $v - v_*$ and $v' - v'_*$. Maxwell computed it for hard spheres ($\widetilde{B} \sim |v - v_*| \sin\theta$) and for inverse power forces: in the latter case the kernel factorizes as the product of $|v - v_*|^\gamma$ with a function $\widetilde{b}(\theta)$; Maxwell showed that if the force is repulsive, proportional to r^{-s} (r the interparticle distance), then $\gamma = (s - 5)/(s - 1)$ and $\widetilde{b}(\theta) \simeq \theta^{-(1+\nu)}$ as $\theta \to 0$, where $\nu = 2/(s - 1)$. In particular, the kernel is usually *nonintegrable* as a function of the angular variable: this is a general feature of long-range interactions, nowadays called "noncutoff property". Maxwell further noticed that the inverse power $s = 5$ leads to simplified formulas, which could lend themselves to more explicit computations.

He went on to discuss possible boundary conditions. Particles arriving at a point x in the boundary with velocity v may be assumed to acquire a new velocity $R_x v$, determined either by the model of specular reflection ($R_x v = v - 2v \cdot n_x\, n_x$, where n_x is the unit ingoing normal vector at x), or the model of bounce-back reflection ($R_x v = -v$). Either way, the boundary condition reads $f(x, R_x v) = f(x, v)$. In a more sophisticated models, particles are assumed to be absorbed by the boundary and reemitted at given rate, say a Gaussian distribution whose dispersion is dictated by the temperature of the wall:

$$f(x,v) = \rho_-(x)\, M_w(v), \quad v \cdot n_x > 0,$$

where

$$\rho_-(x) = \int_{v \cdot n_x < 0} f(x,v) |v \cdot n_x| \, dv, \qquad M_w(v) = \frac{e^{-|v|^2/(2T_w)}}{2\pi T_w^2}.$$

Maxwell, understanding that the boundary behavior of a gas was a very complex matter, also considered combinations of the above models — nowadays called *Maxwell conditions*.

In order to find the stationary solutions, that is, time-independent solutions of (2), he identified certain particular *hydrodynamic* solutions, which make the collision contribution vanish. These are *Gaussian distributions with a scalar covariance*:

$$f(v) = \frac{\rho\, e^{-\frac{|v-u|^2}{2T}}}{(2\pi T)^{3/2}},$$

where the parameters $\rho > 0$, $u \in \mathbb{R}^3$ and $T > 0$ can be identified respectively as the density, mean velocity, and temperature of the fluid. These parameters can be fixed throughout the whole domain (providing in this case an *equilibrium distribution*), or depend on the position x and time t; in both cases collisions will have no effect. It was absolutely remarkable that Maxwell could recover in this way the Gaussian distributions which already played a central role in probability theory; in the context of kinetic theory, these distributions are thus called **Maxwellian**.

Maxwell went further and made the connection with classical fluid mechanics, in which the equations are expressed in terms of ρ, u and T. He suggested that one could go from the kinetic equations to hydrodynamic equations in certain regimes, and make some predictions on the hydrodynamic behavior from the kinetic theory. Let us quote for instance two counterintuitive effects which Maxwell guessed through kinetic formalism. One is that the viscosity of a low-density fluid hardly depends on density. Another one is the paradoxical "thermal creep effect": a gas which has a temperature gradient parallel to a fixed wall will have a tendency to flow from cold to hot near the wall.

A few years after Maxwell's masterpiece was published, Boltzmann rewrote and deepened this theory, completing the foundations of modern physical kinetic theory.

2. Boltzmann's entropy and collisional relaxation

The word "entropy" was coined by Clausius to designate a certain quantity associated with the tendency to relax or achieve equilibrium. The properties of entropy in relation to exchanges of heat and energy were established empirically, in particular the formula for infinitesimal variation of entropy: $dS = \delta Q/T$ (variation in entropy is proportional to the exchanged heat divided by the temperature). In this vein came the well-known *second law of thermodynamics*, which states that in an isolated system entropy can never decrease. Even though there were rules to compute the entropy of an equilibrium system, the interpretation of that quantity remained somewhat elusive, and the second law was considered more or less as an axiom.

That changed radically with Boltzmann's contribution (1872–1877). In one of the most dramatic events in the history of statistical physics, Boltzmann introduced

• a general *mathematical definition* of entropy: it is the logarithm of the volume of microscopic states which are compatible with the (observable) macrostate. This is the celebrated Boltzmann formula:

$$S = k \log W, \tag{3}$$

where k is Boltzmann's constant (a notation introduced by Planck, who was the first to estimate its value), and W stands for the volume of microscopic states. Here the volume may be computed with some natural measure on the phase space, which may be discrete or continuous, depending on the cases.

• a *practical formula* for computing the entropy of a kinetic system: if $f(x, v)$ is the distribution function, then $S = -\iint f \log f \, dx \, dv$. This is derived from Boltzmann's formula (3) through a discretization procedure; it can also be seen as an infinite-dimensional analogue of Liouville's volume measure.

• a *theorem* showing that the entropy of a gas which obeys the equations discovered by Maxwell can never decrease.

More precisely, Boltzmann's H Theorem states that for a rarefied gas modelled by a kinetic distribution $f(t, x, v)$, governed by the Boltzmann equation with appropriate boundary conditions, the functional $H = -S$ satisfies (i) $dH/dt \leq 0$; (ii) $dH/dt = 0$ if and only if $f(t, x, v)$ is a Maxwellian distribution with possibly variable parameters ρ, u, T. Such a distribution can be called **hydrodynamic** since it only depends on hydrodynamical quantities.

In fact an exact formula can be given for the entropy production: for, say, specular reflection,

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \int D\big(f(t, x, \cdot)\big) \, dx,$$

(4)

$$D(f) = \frac{1}{4} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{S}^2} \widetilde{B}(v - v_*, \omega) \, \big(f(v')f(v'_*) - f(v)f(v_*)\big) \log \frac{f(v')f(v'_*)}{f(v)f(v_*)} \, dv \, dv_* \, d\omega.$$

Establishing this expression was one of Boltzmann's motivations to rewrite Maxwell's kinetic equation (expressed in some sort of weak formulation) into the modern form (1).

The H Theorem implies in particular that stationary states have to be hydrodynamic at all times; Boltzmann shows that in the absence of special symmetries this forces the distribution to be spatially homogeneous. This homogeneous Maxwellian is the only equilibrium state, and in this case the entropy coincides with Clausius's entropy.

Boltzmann's beautiful proof was a giant conceptual leap. First, it provided a general definition of entropy, covering nonequilibrium situations: entropy in the Boltzmann theory can be considered as the typical uncertainty which remains in the state of a random particle taken in the system. Then, Boltzmann showed that the second law of thermodynamics could be considered as a logical consequence of fundamental postulates, instead of being accepted as a God-given fact. It also showed that equilibrium thermodynamics could in principle follow from nonequilibrium dynamics, and identified entropy increase, together with a scale separation, as the major factors for the emergence of hydrodynamics.

Still the most dramatic consequence of Boltzmann's work was the display of the *irreversibility* contained in (1), even though that model was derived from Newton's reversible equations of motion. This emergence of irreversibility in the many-particle limit would trigger a heated controversy involving preeminent scientists such as Poincaré, Loschmidt and Zermelo. It is still considered as the classical explanation of the irreversibility of time at the macroscopic level of description, in spite of the reversibility of the full scale evolution.

Entropy increase in the Boltzmann equation shows that particle configurations, in a way, always evolve from unlikely to likely, from exceptional to typical. Thus information is continuously lost (the gas may have started in a very interesting, exceptional configuration, but soon it becomes quite uninteresting). It hence flows from the macroscopic observable degrees of freedom to the microscopic invisible ones. This loss of information can be related to the separation of scales inherent in the derivation of the Boltzmann equation: at each encounter between particles, parameters of the collision (say the orientation of the colliding pair) are invisible because they occur on a scale much finer than the spatial scale, so collisions are treated as perfectly localized and the impact parameter is treated probabilistically. The inexorable increase of entropy can also be attributed to the huge numbers involved in the computation of probability: $N = 10^{20}$ is a large number, but when one enumerates possible configurations, this number appears in a combinatorial way, leading to numbers such as $2^{10^{20}}$ which are so large as to defy any attempt of human grasp.

After being imported to mathematics, entropy was extraordinarily successful in helping to solve problems, both related and unrelated to kinetic theory. It was rediscovered by Shannon when he was building the theory of communication, and it still plays a central role in information theory. It is the basis of Sanov's formula in the theory of Large Deviations for empirical measures. It was a key concept behind Nash's proof of the celebrated De Giorgi–Nash theorem of continuity of solutions of nonsmooth divergence parabolic equations. It lies at the core of the theory of logarithmic Sobolev inequalities, introduced by Nelson and Gross as an infinite-dimensional replacement for Sobolev inequalities. It was one of the key technical tools in the theory of probabilistic hydrodynamic limits initiated by Varadhan and his colleagues and students; the role of entropy in hydrodynamical limits was further reinforced with Yau's relative entropy method. It was an important tool in the DiPerna–Lions theory of weak solutions of the Boltzmann equation. Much further away from physics, entropy was adapted by Voiculescu in the context of free probability to help solve elusive problems from the theory of Von Neumann algebras.

One of the reasons for the ubiquity of entropy is its extensivity property,

$$S(f \otimes g) = S(f) + S(g),$$

which is natural from the physical point of view (informations associated to two independent variables should add up), and implies an additive dependence on dimension, eventually leading to dimension-independent inequalities.

Nowadays, entropy is still actively being used to develop new tools and techniques. To quote just a few recent works in which it played a crucial role: the infinite-dimensional interpolation inequalities of Otto–Villani; the amazing solution of the Poincaré conjecture by Perelman; the theory of synthetic Ricci curvature bounds by Lott–Sturm–Villani; the solution of Kac's problem of propagation of chaos for the spatially homogeneous Boltzmann equation, by Mischler–Mouhot (after preliminary work by Carlen, Villani, and others).

3. Landau's damping and collisionless relaxation

During the first forty years after its birth, kinetic theory mostly focused on the effect of collisions, which are brutal encounters between particles. A notable development, starting with Lorentz in 1905, was the introduction of transport equations to describe the motion of particles wandering in an array of scattering obstacles, such as beams of electrons or neutrons in metals. The resulting linear collisional equations would later have considerable importance in nuclear physics.

However, around that time it was also realized that in various cases the collective effect of particles on each other is more important than collisions, and leads to a rich variety of behaviors. This was the beginning of mean-field (noncollisional) theory.

Mean-field theory was first introduced in galactic dynamics. In 1915, Jeans discussed the use of the Boltzmann equation to model the evolution of galaxies over millions or billions of years, with each star considered as a particle. He came to the conclusion that, as a first approximation, collisions can be dropped out and one can model the interaction by letting each particle feel a force field which is the resultant of all other particles.

Then the story was repeated in plasma physics, which is primarily governed by the Coulomb interaction between electrons. For such interactions, the collision kernel can be computed explicitly, but leads to a diverging collision operator. In 1936 Landau remedied this situation by replacing the Boltzmann operator by an integro-differential collision operator,

$$Q_L(f,f) = \frac{\log \Lambda}{2\pi\Lambda}\, \nabla_v \cdot \left(\int_{\mathbb{R}^3} a(v-v_*)\big[f(v_*)\nabla f(v) - f(v)\nabla f(v_*)\big]\, dv_* \right), \tag{5}$$

where Λ (the plasma parameter) is a large constant, and $a_{ij}(z) = (\delta_{ij} - z_i z_j/|z|^2)/|z|$. But Vlasov in 1938 pointed out that the effect of collisions can be disregarded as well, except in the long-time analysis, so that the distribution of electrons is mainly subject to the force generated by electrons, coupled with the electrostatic equations of Maxwell.

Be it for galaxies or for plasmas, in both situations the basic evolution equation is

$$\begin{cases} \dfrac{\partial f}{\partial t} + v\cdot\nabla_x f + F[f]\cdot\nabla_v f = 0, \\ F[f](t,x) = -\displaystyle\iint_{\Omega\times\mathbb{R}^3} f(t,y,w)\nabla W(x-y)\, dy\, dw, \end{cases} \tag{6}$$

where Ω is the position domain, and W is the interaction potential, assumed even. This equation, which comes with various admissible boundary conditions, is usually called the **Vlasov equation**, although a historically more appropriate name might be the collisionless Boltzmann equation. The most important cases are when W is the fundamental solution of the Poisson equation $-\Delta W = \delta$ (Coulomb interaction, positive type) or $\Delta W = \delta$ (Newton interaction, negative type). Here the notion of "type", coming from harmonic analysis, refers to the sign of the Fourier transform of the fundamental solution. These cases give rise to the so-called **Vlasov-Poisson equation**.

With collisions absent, the most striking features of the Boltzmann equation disappear: thus (6) does not possess any meaningful Lyapunov functional, except that its solutions satisfy the conservation of the energy

$$E = \frac{1}{2}\iint\!\!\iint f(x,v)\, f(y,w)\, W(x-y)\, dx\, dy\, dv\, dw + \iint f(x,v)\frac{|v|^2}{2}\, dx\, dv$$

and of all nonlinear functions of the density $\iint C(f)\, dx\, dv$. In particular, the entropy is constant.

Also in contrast with (1), equation (6) has a surprisingly large collection of steady states: in the absence of external field or boundaries, this includes in particular homogeneous distributions $f^0(v)$, but also many inhomogeneous periodic stationary solutions.

All this seems to oppose the idea that solutions of (6) would display a definite large time behavior. It thus came as a huge surprise when Landau, in 1946, showed that the linearized analysis of (6) for Coulomb interaction led to the exponential decay of perturbations for a large class of equilibria and perturbations. (For instance, if both the equilibrium and the perturbation are analytic distributions, and the equilibrium has only one maximum in dimension 1, or depends only on $|v|$ in dimension 3). This effect has been dubbed **Landau damping**; because it seemed to find irreversibility where everything looked reversible, it struck contemporary physicists probably as much as Boltzmann's discovery of the collisional increase of kinetic entropy — even though its physical impact was much more restricted than the H Theorem.

In contrast with Boltzmann's H Theorem which is genuinely nonlinear, Landau's damping was based on a linearized computation. The results of Landau were refined and extended by a number of physicists, including O'Neil, Penrose, Backus, Maslov, Fedoryuk and others. When experiments became accessible, Landau's computations were verified with a good degree of accuracy, and Landau damping became one of the cornerstones of modern classical plasma physics. It was later exported to galactic dynamics by Lynden-Bell.

Having been discovered by mathematical computation, Landau damping has led to considerable speculation about its driving mechanism, and generated a number of misleading ideas. The most convincing interpretation is that collisionless transport phenomena involve a *mixing* of the distribution function via very fast kinetic oscillations, which in the stable case have a tendency to wipe out inhomogeneities.

Even though the collisional and the collisionless analysis are both idealizations, they constitute the basis of most of our understanding of kinetic theory now. They can interact with each other: for instance the tendency to homogenize faster than expected can enhance the impact of diffusion or collision on relaxation phenomena.

At the same time as all this analysis was being developed, the mean-field analysis started to be applied to a number of situations outside kinetic theory, both in equilibrium and nonequilibrium systems. In particular, in a famous discussion of turbulence, Onsager studied the incompressible 2-dimensional Euler equation in vorticity form as a mean-field system of "vortices", presenting many similarities with collisionless kinetic theory.

4. Driving problems

The development of kinetic theory, and Boltzmann's very influential book, quickly attracted the attention of mathematicians, starting with Hilbert, who formulated his Sixth Problem (related to items (I) and (IV) below) under the inspiration of Boltzmann. Hilbert himself did some early mathematical study on the Boltzmann equation, followed by Carleman in the thirties, then Grad and Kac in the fifties. These works were focused on Boltzmann's collision operator.

As the theory of partial differential equations was making progress, the analysis of the tricky transport operator $v \cdot \nabla_x$ began, in relation with other operators. This can be traced back to Kolmogorov's work on the fundamental solution of the kinetic Fokker–Planck equation.

It took longer for the Vlasov (collisionless) theory to make its way into mathematics; this task was undertaken only at the end of the seventies in Russia, with Arsen'ev and Dobrushin. Soon after, Braun, Hepp and Neunzert followed in the Western world.

A number of problems emerged from these works, at the interface of mathematics and physics; they have been driving the field for decades, and triggered far-reaching developments. For the most part, these problems fall into five general themes which are all related:

(I) *Derivation from first-law principles:* Starting from fundamental equations such as the Newton laws or certain simple diffusive microscopic models, derive kinetic statistical equations. To derive collisional equations, one is often led to justify, directly or indirectly, Boltzmann's **chaos assumption**: pre-collisional configurations are uncorrelated. Chaos assumptions also play an important role in noncollisional models, and more generally in the derivation of any deterministic equation on the distribution function.

(II) *Cauchy problem and qualitative analysis:* Starting from an initial datum which satisfies certain assumptions of smoothness and decay at large velocities, prove that the solution is well-behaved, make precise the way in which it solves the kinetic equation, and establish whether bounds of smoothness, large-velocity decay and strict positivity are preserved in time. Is there regularization, or at least decay of the amplitude of singularities? In many situations, the understanding of the Cauchy problem precludes the rigorous derivation.

(III) *Long-time behavior:* Starting close to some equilibrium, does the solution remain close to the equilibrium for all times (orbital stability)? does it converge to the equilibrium, or to some equilibrium (dynamical stability)? Starting far from equilibrium, does it converge to some equilibrium, and can one identify the latter? Are there mixing properties (for instance oscillations developping in time and leading to some weak convergence mechanism)?

(IV) *Relation to other models:* Can one replace, in suitable asymptotic regimes, the kinetic equations with reduced models, such as compressible or incompressible hydrodynamic equations (in the hydrodynamic limit, that is when the mean free path becomes negligible with respect to the spatial scale), or boundary layer equations (when one looks very closely near an interface)? Can one couple kinetic models with other models? or reduce the description by using a multiscale analysis? Can one use the kinetic equations to retrieve observable properties of fluids, such as thermodynamic laws of pressure, viscosity dependencies, phase transition diagrams? An important limitation of the classical theory of Boltzmann is that it only covers perfect fluids, that is, with a pressure law proportional to the product of the density by the temperature.

(V) *Numerical simulation:* Can one devise numerical methods which are fast and accurate? particulary suitable to predict the value of a given quantity? satisfying given constraints? Can one prove that these schemes do converge to the solutions of the corresponding kinetic equations? The latter problem may be strongly related to the derivation problem, because a number of schemes are based on particle simulations. It is also obviously related to the analysis of the Cauchy problem.

In view of their archetypical nature and their relation to fundamental issues such as the arrow of time, the basic equations of kinetic theory have aroused an interest among theoretical physicists, going far beyond the range of application of these models.

We shall describe some of these problems in more detail after discussing the models more precisely.

5. The many models of kinetic theory

Initially, kinetic theory was devised to model rarefied gas dynamics (Boltzmann equation), galactic dynamics (mean field model with Newton interaction), and ideal plasma dynamics (mean field model with Coulomb interaction). All three domains of application are important: for instance, the Boltzmann equation is crucial in high altitude aerodynamics, since the upper atmosphere is not dense enough for the laws of hydrodynamics to apply satisfactorily. More recently, Boltzmann equations have been found useful in the modelling of nanofluids.

The kinetic formalism is versatile and its range of application has been widened considerably beyond these situations. The many resulting variants of the basic equations can be grouped in several categories:

• Classical models with an interaction kernel derived from various molecular interactions, or modified from those which come from the laws of classical physics. In particular, since Grad's work on the properties of the linearized Boltzmann operator, one often truncates small deviation angles to ensure the angular integrability of the collision kernel. Under this assumption of *angular cutoff*, the collision operator can be split into two parts:

$$Q(f,f) = Q^+(f,f) - Q^-(f,f) = \int\int \widetilde{B}(v-v_*,\omega) f(t,x,v')\, f(t,x,v'_*)\, dv_*\, d\omega$$
$$- \int\int \widetilde{B}(v-v_*,\omega) f(t,x,v)\, f(t,x,v_*)\, dv_*\, d\omega,$$

respectively called the gain and loss parts of the operator. By contrast, a kernel which is nonintegrable in the angular variable is called "noncutoff", which corresponds to long-range interactions. Moreover, the interaction is called *hard* if the corresponding collision kernel is proportional to a positive power of the relative velocity, and *soft* if the kernel is proportional to a negative power of the relative velocity; in between lies the *Maxwellian* case where the kernel does not depend on the relative velocity. Hard, Maxwellian and soft potentials often enjoy distinctive properties. A particular case is that of *hard spheres*, in which the kernel is just proportional to $|\langle v - v_*, \omega\rangle|$.

• Models obtained from large particle systems by putting emphasis on various interactions according to physical conditions (density, strength of interaction, etc.) Popular and versatile models are the Fokker–Planck equations, going back to the thirties, which describe the evolution of a crowd of particles undergoing stochastic diffusion and deterministic drift. Systematic derivation of statistical models for particle systems goes back to Bogolyubov. It is especially in the field of plasma physics that this approach has led to a large number of variants. The Balescu–Lenard and Vlasov–Fokker–Planck–Landau equation are among the best known of these models: they incorporate both mean-field and collisional interactions, with collision operators that behave like nonlinear diffusions in velocity space, while bearing a resemblance to the integral and bilinear structure of the Boltzmann operator – recall (5). These models try to reduce the great variety of processes going on in plasmas to tractable equations.

• Linear models describing the interaction of a particle system with a given (deterministic or random) environment. In this category belong in particular the linear Boltzmann equation describing the scattering of particles by a cloud of randomly located obstalces; the archetypal kinetic equation of Fokker–Planck type, studied by Kolmogorov,

$$\partial_t f + v \cdot \nabla_x f - \Delta_v f = 0; \tag{7}$$

the equations of electron transport which are useful in neutronics and in semiconductor theory; and more generally, a variety of equations describing combinations of transport, scattering, diffusion, etc.

• Spatially homogeneous models, in which one studies solutions which do not depend on the position variable but just on the kinetic variable. This leads in particular to the spatially homogeneous Boltzmann equation $\partial_t f = Q(f,f)$, whose study is very well developed, and allows the understanding of fine properties of the collision operator. Additional structure can be achieved by restricting the setting even further, for instance considering only Maxwellian interactions, in which the collision kernel $\widetilde{B}(v-v_*,\omega)$ depends only on the deflection angle θ. The dimension can also be reduced, leading for instance to Kac's one-dimensional caricature of a Boltzmann gas, in which velocities are one-dimensional and the conservation of energy has been kept but the conservation of momentum has been dropped.

• Linearized equations, obtained by looking at first-order perturbations. For the Boltzmann equation near a homogeneous Maxwellian $M = M(v)$, the linearization is

$$\partial_t h + v \cdot \nabla_x h = Q(h,M) + Q(M,h),$$

which is often further transformed by conjugation with a multiplication operator. For the Vlasov equation near a homogeneous equilibrium $f^0 = f^0(v)$, this is

$$\partial_t h + v \cdot \nabla_x h + F[h] \cdot \nabla_v f^0 = 0.$$

In both cases, spectral properties depend significantly on the interaction potential, and have been the object of numerous studies. Many variants of these archetypal models are available.

• Delocalized models, in which particles are allowed to have a nonnegligible interaction range. While this procedure is logically inconsistent with the many-particle limit, it does produce some useful equations, such as the Povzner equation and especially the Enskog equation, used in the description of granular matter.

• Models incorporating different physical laws: inelasticity (replacing the energy conservation by a dissipation law; this approach is especially important in the modelling of granular matter); quantum physics (either by modelling quantum phenomena in the interaction terms, thus leading to Boltzmann–Bose or Boltzmann–Fermi models for bosons or fermions collisions, or by keeping a classical description of collisions but incorporating quantum effects in the computation of the cross-section); relativity (either by incorporating the geometry of special relativity in the laws of interaction, or by coupling a kinetic equation to the constitutive equations of general relativity); and so on. In relation to relativity, it should be noted that the Einstein equations of general relativity cannot "stand on their own" unless one studies the vacuum: these equations need to be coupled to an evolution equation for matter, satisfying certain conditions. Since the pioneering works of Choquet–Bruhat on the Cauchy problem in general relativity, the Vlasov equation has been studied in this context, giving rise to the so-called Vlasov–Einstein model.

• Coagulation-fragmentation models incorporating crude modellings of chemical reactions, drop formation from molecules via larger and larger gatherings, gelation problems… The Smoluchowski equation is one of the most popular in this respect.

• Discrete velocity models, lattice models, devised to simplify the geometry of collisions and the phase space, for instance for numerical simulations.

• Models appearing in various other physical contexts, such as interactions between waves in models of weak turbulence. Another example is the collisionless kinetic equation obtained by application of the Wigner transform to the Schrödinger equation.

• Kinetic equations in interaction with other physical phenomena: coupling of radiative transfer and hydrodynamics (in astrophysics or nuclear physics), of particles and hydrodynamic fluids (in sprays for instance), etc.

• Phenomenological models for various interaction phenomena that are difficult to classify, or evade precise physical modelling: crowds, traffic, disease transmission, sexual reproduction…

6. The many mathematical faces of kinetic theory

Modern kinetic theory enjoys an enviable place within the whole mathematical landscape: standing on top of it, the curious observer can view most of the regions of analysis, as well as significant territories of probability and geometry. In the past thirty years this theory has been interacting with many other fields, and displayed a number of sophisticated developments. At the same time, it progressed from a rather confidential status to reach the spotlights.

Compared to other fields of mathematical physics, some of the particular features of kinetic theory are

- the presence of two variables (position and velocity);
- the omnipresence of large velocities, which cannot be truncated in the model;
- the degeneracy of most equations in the spatial variable;
- the intricate geometry of collisions;
- the fact that kinetic theory is at a crosspoint between several areas of modelling;
- the interplay of deterministic and chaotic behavior.

With this in mind, here are some of the main mathematical tools and tendencies in kinetic theory:

• *Spectral theory:* The linearized Boltzmann equation was of the first model cases of study for integro-differential operators. Some of the important notions here are spectral gap estimates, the Fredholm alternative, self-adjointness, localization of essential spectrum, compact perturbations, compactness of the resolvent, accretivity…

• *Nonlinear analysis of the Cauchy problem:* Tools involve a priori estimates (starting with the mass, energy and entropy controls), the Cauchy–Kowalevskaya theorem (especially in the short-time derivation of the Boltzmann equation, or early theories for the Vlasov–Poisson equation), Kolmogorov–Nash–Moser perturbation techniques, the Moser scheme (especially for the spatially homogeneous Boltzmann equation for long-range interactions, which has dissipative features), weak compactness theorems (in particular in the DiPerna–Lions theory of weak solutions), Sobolev trace theorems, weighted functional spaces of Lebesgue, Sobolev, analytic and Gevrey type. Bilinear and trilinear estimates with a strong input from harmonic analysis were recently developed for the study of long-range interactions, in which the collision operator behaves more or less like a nonlinear fractional derivation. This list must also include the nonlinear changes of variables used by DiPerna and Lions in their notion of "renormalized" solutions; and the "gliding" regularity analysis (regularity obtained after composing the function with a transport equation) for the study of fast oscillations of the Vlasov equation in large time, etc.

• *Harmonic analysis:* Fourier analysis, either in the position or in the velocity variable, has played a crucial role in various parts of the theory. Most notably the analysis of the spatially homogeneous Boltzmann equation with Maxwellian interactions (for which the Fourier transform of the collision operator is particularly simple and tractable), the long-time perturbative analysis of the nonlinear Vlasov equation (Landau damping being analyzed mode by mode), the regularity of the gain part of the Boltzmann operator (which is more regular, by a fractional amount, than the density function); and velocity-averaging estimates. The latter are intended to answer the following type of question: given an equation like $v \cdot \nabla_x f = g$, with certain regularity information on f and g, show that if a smooth test function φ is given, then $\int f(x,v)\,\varphi(v)\,dv$ enjoys more regularity than can be predicted from just the regularity of f. This point of view was extremely fruitful in modern studies of the Cauchy problem, and is based mainly on Fourier or X-ray transforms.

• *Entropic inequalities:* The analysis of the long-time behavior of collisional kinetic equations naturally leads to the study of inequalities relating Boltzmann's entropy and its rate of production. Kac and McKean were the first to address these issues from the mathematical point of view, and made the connection with information theoretical inequalities, involving for instance the Fisher information. A central topic in the field came to be known as *Cercignani's conjecture:* is it true that, under certain conditions of normalization or regularity, the entropy production (4) satisfies the functional inequality $D(f) \geq K\,[S(M) - S(f)]$, where M is a Maxwellian distribution? This problem, the entropic variant of a spectral gap inequality, has led to unexpected and rich developments related to logarithmic Sobolev inequalities, or the Shannon–Stam and Blachman–Stam inequalities.

• *Semigroup arguments:* At the end of the nineties, semigroup arguments made their way in kinetic theory, either through the use of auxiliary diffusion equations, or by the second variation method introduced by Bakry and Émery in their study of logarithmic Sobolev inequalities. The ideas of Bakry and Émery were adapted in the context of partial differential equations by Toscani, Arnold, Markowich and others; a large body of works followed on the entropic analysis of the convergence to equilibrium in long time, both for linear and nonlinear models in kinetic theory, especially of Fokker–Planck type. Some of the key concepts are the Γ_2 calculus, curvature-dimension inequalities, and dissipation of entropy production (i.e. second time-derivative of the entropy).

• *Specific techniques for degenerate operators:* Kolmogorov in the fifties, and Hörmander in the sixties, founded the theory of **hypoellipticity**, according to which certain degenerate opeartors, like $-v \cdot \nabla_x + \Delta_v$, generate a regularizing semigroup. This situation, which occurs frequently in linear dissipation kinetic models, is often treated by commutator estimates. The more recent theory of **hypocoercivity** deals with the time decay of semigroups generated by degenerate operators, typically of the form $T + \Lambda$, where Λ is coercive in some appropriate subspace, and T is skew-symmetric. Paradigmatic examples are $-v \cdot \nabla_x + \Delta_v - v \cdot \nabla_v$ in $L^2(M\,dx\,dv)$ and $-v \cdot \nabla_x + \Pi_M - \mathrm{Id}$ in $L^2(M\,dx\,dv)$, where M is a Gaussian and Π_M is the orthogonal projection on constant functions.

• *Qualitative studies of solutions:* The Vlasov equation is of hyperbolic nature, but the Boltzmann equation is of mixed hyperbolic/parabolic type in some sense; a number of works and techniques have been devoted to the studies of the qualitative behavior of solutions, including regularization, propagation or decay of singularities (often studied in Sobolev spaces), wave patterns (in connection with systems of conservation laws and compressible Navier–Stokes equations), harmonic analysis, pseudo-differential operators, Littlewood–Paley analysis, Radon transform, quantitative uncertainty principles, self-similar ansatz, concentration analysis. . .

• *Singular limits:* These limits, in which a term of the equation is enhanced by a diverging coefficient, are studied via ansatz, expansions, spectral theory, ergodic theory, etc. They appear in particular in connection with (a) inviscid or viscous hydrodynamic limits, in which the Knudsen number (ratio of the mean free path to the typical length) goes to 0, typically leading to an enhanced collision operator, $\varepsilon^{-1}Q(f,f)$; (b) the homogenization of transport models, typically leading to an enhanced transport term, $\varepsilon^{-1}v \cdot \nabla_x$ or $\varepsilon^{-1}F \cdot \nabla_v$; (c) high-frequency semiclassical limits of Schrödinger equations, via the Wigner transform; (d) small mass ratio limits, for instance in plasmas where electrons are much lighter than nuclei.

• *Differential geometry:* Curved phase spaces appear naturally in the relativistic kinetic theory, either through the rules of collisions between particles, or because the system is considered in a Lorentzian ambient space.

• *Calculus of variations:* When stability is not ensured by a Lyapunov functional such as the entropy, stability issues can be very tricky. Then convexity properties can be crucial in studying the dynamic stability of particular equilibria which are energy minimizers. This approach, introduced in hydrodynamics in the sixties, was systematically used from the eighties on in the theory of the Vlasov–Poisson equation, with the help of notions of concentration-compactness, rearrangement, etc.

• *Many-particle techniques:* The quest for a rigorous foundation of the Boltzmann and Vlasov equations from particle systems has led to the analysis of many-particle systems, obeying the fundamental laws of classical or quantum mechanics, in the limit where the number N of particles diverges to infinity. Then the microscopic equations depend on all positions and velocities, say $(x_1, v_1), \ldots, (x_N, v_N)$. The problem can be set in terms of the likely behavior of the empirical distribution, say $\widehat{\mu}^N = N^{-1} \sum_{i=1}^N \delta_{(x_i,v_i)}$; or in terms of the limit behavior of the first-particle marginal of an N-particle distribution f^N, satisfying the N-particle Liouville equation $\partial_t f^N + \sum v_i \cdot \nabla_{x_i} f^N - \sum_{i \neq j} \nabla W(x_i - x_j) \cdot \nabla_{v_i} f^N = 0$, in various asymptotic regimes where time, space, mass and strength of interaction may be rescaled. Popular scalings are the Boltzmann–Grad limit pioneered by Grad, Cercignani and Lanford, leading to the nonlinear Boltzmann equation; and the mean-field limit established by Braun, Hepp, Dobrushin and Neunzert for smooth interactions, leading to the Vlasov equation. Famous variants are: the probabilistic approach of Kac, in which the deterministic Newton equations are replaced by a phenomenological, stochastic microscopic model; and the derivation of the linear Boltzmann equation, for a so-called Knudsen gas, pioneered by Gallavotti. Key concepts in this field are notions of molecular chaos (asymptotic independence of particles, i.e. low correlations), perturbative series, particle histories, functional inequalities in infinite dimension, quantitative laws of large numbers, central limit theorems, and orthogonal polynomials. For quantum particle systems, density matrices and Wigner transforms play a crucial role.

• *Numerical analysis:* The simultaneous presence of very diverse terms in the equations, the high dimensional phase space, the complexity of collisions, the presence of large velocities and small densities, the difficulty of accurate experiments — all of this has made numerical simulation of kinetic models a challenging area. Transport phenomena are most often simulated with the help of the methods of characteristics, that is, following particles in phase space; but the reconstruction of the density from one time step to the other leads to many subtleties, since the number of particles used in the simulation is always much smaller than the actual number of particles, and since particle trajectories do not preserve grids or other discretizations of the phase space. Collisional phenomena are tricky to compute and were initially handled by stochastic methods, based on particle systems obeying more or less realistic interaction rules. These schemes were founded by Bird in the sixties, and remained dominant for more than forty years. It is only in the last decade that the progress of algorithms and computer power made the more accurate deterministic methods cost-competitive, at least in certain situations. Keywords here are the splitting method, the Monte Carlo simulation, consistency analysis, Lagrangian and semi-Lagrangian methods, spectral analysis, Fourier transform, fast Fourier transform algorithm, finite elements, lattice simulation, conservative schemes, adaptive grids, etc. Specific methods were developed by Cheng & Knorr, Sone, Aoki, Babovsky, Neunzert, Wagner, Degond, Bobylev, Rjasanow, Sonnendrücker, Pareschi, Filbet, and many others. In aeronautics, in astrophysics, in plasma physics, this is an enormous amount of literature which is hardly touched on in this review.

Let us conclude this list with two subjects which were partly motivated by kinetic theory, but where the main impact was in other parts of mathematics.

• *Ordinary differential equations with rough coefficients:* The classical theory of ordinary differential equations, say $\dot{x} = \xi(t, x)$, requires continuity of ξ for the local existence of a flow, and Lipschitz regularity of ξ for the local uniqueness and continuous dependence. This requirement of Lipschitz regularity is often a strong restriction in applications to partial differential equations, especially when ξ depends on the solution and its regularity is a priori unknown. As a by-product of their studies of the Cauchy problem in kinetic theory, DiPerna and Lions came up with a theory of ordinary equations which provides local existence and uniqueness for *almost every* initial data, under a more lenient assumption of Sobolev regularity (for instance $\xi \in W^{1,1}_{loc}$, $\operatorname{div} \xi \in L^\infty$ and some growth condition at infinity). The original proof was based on the analysis of the transport equation $\partial_t f + \xi \cdot \nabla f = 0$ and the renormalization technique; more recently, the theory was refined by Ambrosio, De Lellis and others, to include the limit case of bounded variation regularity, and to provide alternative, trajectorial proofs. This theory has been used in various types of partial differential equations, such as hyperbolic systems of conservation laws.

• *Optimal transport and metric geometry:* In the seventies Tanaka showed that the spatially homogeneous Boltzmann equation with Maxwellian interactions is contracting for the Wasserstein (optimal transport) distance

$$W_2(\mu, \nu) = \left(\inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - v_*|^2 \, d\pi(v, v_*) \right)^{1/2},$$

where μ and ν are two probability measures on $\mathbb{R}^d$, and the infimum is over all joint probability measures $\pi(dv\,dv_*)$ with marginals μ and ν. After a period where these connections sank more or less into oblivion, the links between optimal transport and kinetic equations were renewed in the nineties, and led to various uniqueness and stability results. Later the interplay between optimal transport and Boltzmann's entropy played a key role in the theory of nonsmooth Ricci curvature.

7. Landmarks

Below is a list of 50 striking works, arranged chronologically, which have punctuated progress in kinetic theory and have become classical references. Some of these works are opening a field of research, others are closing one; some are gathered when they are very close in subject. The dates which are indicated are those of publication, sometimes coming a few years after the actual writing.

This list is not intended for comparison of importance and the selection is partly a personal matter. It is biased in favor of theoretical issues, and does not do justice to some subjects which are of great importance in industrial applications, such as neutron transport or coagulation-fragmentation. Similarly, it does not touch the enormous and inventive body of works which have been done in the field of numerical simulations.

This list also partly reflects the history of mathematical kinetic theory: at first it is quite a confidential subject with rare contributions; then a few centers emerge after the Second World War: New York (Courant Institute), Osaka/Kyôtô, Göteborg, Rome, Moscow, Zürich. In the eigthies and nineties, studies of nonlinear problems start to flourish, with the French and Italian schools taking the lead, and the community became quite organized; at the same time new research groups were emerging, most notably in Germany, USA, Austria, Spain, Taiwan, and Canada. The past twenty years have been characterized by the growing and fruitful interplay with other fields of mathematics, by the emphasis on quantitative results and constructive methods, and the renewal of the study of the linearized and perturbative settings.

(1) *Hilbert (1912):* First study of the linearized Boltzmann operator, and first formal expansion of the solution of the Boltzmann equation in powers of the (small) Knudsen number near the hydrodynamic regime.
(2) *Chapman & Enskog (1917):* Systematic formulas for deriving macroscopic transport coefficients from microscopic interactions, and perturbative expansion near the hydrodynamic regime, alternative to Hilbert's.
(3) *Carleman (1932):* First solution to the nonlinear Cauchy problem for the spatially homogeneous Boltzmann equation, for hard spheres; including a qualitative study of lower bound for the density, and a study of the convergence to equilibrium.

(4) *Kolmogorov (1934):* Computation of the fundamental solution of the kinetic Fokker–Planck equation $\partial_t f + v \cdot \nabla_x f = \Delta_v f$, displaying (hypoelliptic) regularity properties.
(5) *Grad (1949):* 13-moment system describing a high order approximation to the hydrodynamic limit of the Boltzmann equation.
(6) *Kac (1954):* Probabilistic foundation of kinetic theory, through a phenomenological stochastic many-particle model of the spatially homogeneous Boltzmann equation; leading to conjectures on quantitative relaxation rates.
(7) *Backus, Penrose (1960):* Mathematical treatment of linear Landau damping, discovered by Landau in 1946, with sharp criteria for stability; statement of the nonlinear damping problem.
(8) *Carleman (1949, 1957), Grad (1963-1965):* Modern spectral theory of the linearized Boltzmann equation with cutoff, for hard interactions, in the homogeneous and inhomogeneous settings.
(9) *McKean (1965):* Probabilistic study of Kac's caricature of Boltzmann equation, through molecular chaos, Fisher information estimates, and quantitative central limit theorem for Maxwell interactions.
(10) *Bird (1966):* First numerical scheme for the stochastic simulation of the Boltzmann equation; an alternative scheme was later introduced by Nanbu (1983).
(11) *Hörmander (1967):* General criteria for hypoellipticity of degenerate diffusion equations; including a precursor of velocity-averaging lemmas.
(12) *Gallavotti (1969):* Derivation of the linear Boltzmann equation from the Lorentz gas, through averaging over a random environment. Many developments have followed, by Pulvirenti, Desvillettes, and others.
(13) *Arkeryd (1972):* Cauchy problem for the spatially homogeneous Boltzmann equation with hard potentials, in weighted L^1 spaces; including weak compactness properties in L^1.
(14) *Tanaka (1973):* Contraction properties of the spatially homogeneous Boltzmann equation with Maxwell kernel, in the Wasserstein W_2 distance.
(15) *Lanford (1974):* Short-time derivation of the Boltzmann equation from deterministic Newton laws.
(16) *Ukai (1974):* Perturbative solutions of the full inhomogeneous Boltzmann equation, based on the spectral theory of the linearized equation.
(17) *Bobylev (1976–1988):* Systematic study of the spatially homogeneous Boltzmann equation with Maxwell interactions, via Fourier transform.
(18) *Braun & Hepp (1977), Dobrushin (1979), Neunzert (1984):* Rigorous mean-field limit for the Vlasov equation with smooth interactions.
(19) *Sznitman (1984):* Propagation of chaos and probabilistic derivation for the spatially homogeneous Boltzmann equation with hard spheres.

(20) *Golse, Perthame & Sentis (1985):* Start of the systematic study of velocity-averaging lemmas in Sobolev spaces, which had been independently introduced by Agoshkov (1984) shortly before.
(21) *Glassey & Strauss (1986):* Cauchy problem for the relativistic Vlasov–Maxwell equation, conditional to a conjectured property of compact support.
(22) *Bony (1987):* New Lyapunov functional for the discrete-velocity inhomogeneous Boltzmann equation in one space dimension; starting point for various Lyapunov functionals for the Boltzmann equation in one space dimension.
(23) *DiPerna & Lions (1989):* Existence and stability of weak solutions ("renormalized solutions") in the large, for the nonlinear inhomogeneous Boltzmann equation.
(24) *Bardos, Golse & Levermore (1991):* Systematic program for the proof of hydrodynamic limits of weak solutions, in particular in the incompressible regime; this program would take 20 years to be completed.
(25) *Lions & Perthame (1991), Pfaffelmoser (1992):* First proofs of existence and uniqueness of classical solutions for the 3-dimensional Vlasov–Poisson equation, by two different approaches.
(26) *Desvillettes (1989), Carlen & Carvalho (1992, 1994):* First lower bounds on the instantaneous rate of entropy production in the Boltzmann equation, through quantitative H Theorem and information theory.
(27) *Desvillettes (1993):* Refined moment estimates for the spatially homogeneous Boltzmann equation, in particular their immediate appearance in the case of hard potentials.
(28) *Lions (1994):* Regularity of the gain term of the Boltzmann collision operator, which is shown to have the structure of a singular integral operator, gaining up to 1 derivative in dimension 3 for smooth kernels.
(29) *Desvillettes (1995):* First evidence of regularization due to long-range interactions in the Boltzmann equation, on a spatially homogeneous caricature; the start of a long series of works on such regularization effects.
(30) *Gérard, Markowich, Mauser & Poupaud (1997) (and Lions & Paul (1993)):* Systematic study of high frequency limits through the Wigner transform, with applications to quantum kinetic theory.
(31) *Erdös & Yau (1998):* Derivation of the linear quantum Boltzmann equation, in the weak coupling limit, for the Wigner distribution of a quantum particle in a random environment.
(32) *Mischler & Wennberg, Lu (1999):* Optimal conditions for the well-posedness of the Cauchy problem of the spatially homogeneous Boltzmann equation with hard interaction and cutoff.

(33) *Carlen, Gabetta & Toscani (1999):* Optimal rates of convergence to equilibrium for the spatially homogeneous Boltzmann equation with Maxwell interaction and angular cutoff (the removal of the cutoff was later obtained in subsequent works involving Wennberg, Dolera & Regazzini).
(34) *Toscani & Villani (1999), Villani (2003):* Sharp entropy production bounds for the Boltzmann equation, solving or nearly solving, depending on assumptions, Cercignani's conjecture; based on semigroup methods, information theory, Landau equation.
(35) *Guo (2002):* First of a series of works using energy methods to work out robust perturbative theories of the Boltzmann equation and other kinetic models.
(36) *Carlen, Carvalho & Loss (2003), Maslen (2003):* Determination of the L^2 spectral gap for Kac's random walk in arbitrarily large dimension, after a uniform lower bound was established by Janvresse.
(37) *Carlen & Lu (2003):* Examples of arbitrarily slow convergence to equilibrium for the Boltzmann equation with Maxwell interactions.
(38) *Bobylev, Gamba, Panferov, Villani (2004):* Moment estimates and Cauchy problem for the inelastic spatially homogeneous Boltzmann equation with hard interactions.
(39) *Alexandre & Villani (2004):* Weak solutions for the spatially inhomogeneous Boltzmann equation without cutoff and limit of predominant grazing collisions in the spatially inhomogeneous case; coming after much progress in the understanding of grazing collisions, by Alexandre, Desvillettes, Villani and Wennberg.
(40) *Liu & Yu (2004):* First works on the pointwise stability and Green function in the Boltzmann theory, and shock wave analysis (motivated by an earlier work of Caflisch), by analogy with systems of conservation laws; the start of a long series of works.
(41) *Golse & Saint-Raymond (2004):* Rigorous proof of the incompressible hydrodynamic limit for weak solutions of the Boltzmann equation; coming after works by Golse, Levermore, Masmoudi and others.
(42) *Desvillettes & Villani (2005):* Quantitative convergence to equilibrium for the Boltzmann equation, far from equilibrium, by entropy methods, under a conjectural condition of regularity bounds.
(43) *Baranger & Mouhot (2005), Mouhot (2006), Gualdani, Mischler & Mouhot (2013):* Optimal rates of convergence for the Boltzmann equation (homogeneous and inhomogeneous), coupling quantitative spectral analysis to entropy methods, conditional to regularity.
(44) *Mischler & Mouhot (2006):* Proof of Haff's law of decay of temperature and self-similar stability in the theory of granular (inelastic) gases.

(45) *Villani (2009):* General criteria for hypocoercivity, both in linear and nonlinear situations.
(46) *Gressman & Strain; Alexandre, Morimoto, Ukai, Xu & Yang (2011):* Construction of smooth solutions for the noncutoff spatially homogeneous Boltzmann equation, for potentials that are hard or not too soft.
(47) *Lemou, Méhats & Raphaël (2011):* Orbital stability of spherical monotone equilibria of the gravitational Vlasov–Poisson equation; the culmination of a long series of works on the stability of Vlasov–Poisson, by Antonov, Wolansky, Strauss, Guo, Rein and others.
(48) *Mouhot & Villani (2011):* Proof of Landau damping for the nonlinear Vlasov equation, near stable homogeneous equilibria, in analytic or Gevrey regularity, via phase mixing and gliding regularity; later adapted by Bedrossian & Masmoudi to the inviscid damping near the Couette flow.
(49) *Mischler & Mouhot (2013):* Significant progress on Kac's program: Relaxation estimates for particle systems, quantitative and uniform in the number of particles, in the limit of the spatially homogeneous Boltzmann equation, using quantitative chaos properties and entropic estimates.
(50) *Escobedo & Velázquez (2013):* Rigorous proof of blow-up (Bose–Einstein condensation) for the quantum spatially homogeneous Boltzmann–Bose equation.

8. Challenges

While kinetic theory has gone a tremendous way, there are still some monster driving problems which are wide open. Here are some of them, gathered according to a few main themes.

8.1. **Cauchy problem, regularity, singularities, finite-time qualitative behavior.** The most important and annoying open problems here are certainly the related questions of regularity and well-posedness for the Boltzmann equation, when no perturbative or spatial homogeneity assumptions are imposed. Parallels could be drawn between this and the Millenium problem on the incompressible Navier-Stokes in dimension 3. For collisionless kinetic equations, even if the Cauchy problem for the Vlasov–Poisson equations has been tamed, other Herculean tasks remain concerning more intricate models. The Cauchy problems for the Vlasov–Maxwell and Vlasov–Einstein equations are of particular interest. Actually, for these two equations, even the perturbative theory is far from well understood. In a completely different direction, the stability of homogeneous solutions remains almost untouched in the theory of the inhomogeneous inelastic Boltzmann equation; the annoying issue here is that nobody has been able to prove the possibility of clustering, well accepted in physics. Finally, much remains to be understood about very soft interactions (when the collision kernel behaves like a large negative power of the relative velocity).

8.2. **Large time behavior.** The entropic relaxation for the Boltzmann equation now seems quite well understood without boundaries, with robust estimates in the large as well as optimal decay from quantitative linearized arguments. The collisionless relaxation in the Vlasov theory on the other hand is understood only near a stable homogeneous equilibrium, and the stability of inhomogeneous stationary solutions, such as the so-called BGK waves, remains a famous open problem. The mathematical theory of instability phenomena is also wide open. The long-time behavior of "typical" data, for instance via a statistical approach, is untouched.

A long-term goal is the combination of entropic and mixing effects in the study of convergence, for instance for the so-called Vlasov–Poisson–Fokker–Planck equations, which combine mean-field mixing and hypoelliptic diffusion.

For dissipative equations with *non-reversible* stationary states, corresponding to nonlocal cancellations in the equation, the long time study is in its infancy. For nonlinear models, even when the existence of equilibria is proven, there is in general no Lyapunov-type approach to the long-time behavior.

8.3. **Meso-macro limits: Hydrodynamic limits.** Huge progress has been made in understanding the hydrodynamic limit of Boltzmann equations, since this problem was expressed by Hilbert more than a century ago. Yet many questions remain unanswered. The incompressible limit is now rather well understood. But this limit is quite specific, and one would like to understand better the more natural *compressible* limit. Examples of problems in this area include the large-time stability of Boltzmann solutions near a smooth solution of compressible Navier–Stokes equations; and the handling of shocks in the large.

As already mentioned, hydrodynamic limits of Boltzmann equation only lead to perfect fluids, unless the equations are modified in a phenomenological way. To retrieve alternative pressure laws from basic principles of classical mechanics, the most natural plan is to go directly from the equations of microscopic many-particle systems to hydrodynamic models, without passing through the mesoscopic scale. This strategy was first made precise in a program sketched by Morrey in the fifties, which proved to be extraordinarily difficult and is still largely open in spite of substantial progress by Varadhan, Yau, Olla and others.

8.4. **Microscopic Derivation.** Derivation of kinetic models from the laws of atomistic matter is also part of Hilbert's 6th problem, and is an emblematic issue both in kinetic theory and in statistical physics.

In the collisional case, the most important open problem is certainly the validity of the Boltzmann–Grad limit for hard spheres in large time (i.e. in time significantly larger than the mean free time) and without any assumption of very small mass. This problem probably includes an understanding of the regularity for the inhomogeneous Boltzmann equation, so it can be considered as a Holy Grail in the field. Another open problem is the low density limit in the case of long-range collisional interactions; in this case, even a short-time result is not established.

In the collisionless case, the main open problem is the rigorous justification of the mean-field limit in the case of Coulomb and Newton interactions. The best results so far have been obtained by Hauray and Jabin around 2007, but still require smoothing or cutoff of the interaction at small scales. A further goal is the understanding of the microscopic derivation of the many involved models which appear in plasma physics, one instance being the Balescu–Lenard equation (for which even the short-time well-posedness is still unclear).

Finally in the case of diffusive kinetic equations, one of the most appealing open problems is the derivation of the heat equation from a set of interacting oscillators, as studied for instance by Rey-Bellet. While preliminary works have established, among other things, the existence of relevant equilibria, the derivation of the heat equation has been understood only in particular cases, with the help of hypoelliptic and hypocoercive tools.

8.5. **The challenge of boundary conditions.** The question of the interaction of gases with boundaries or external forces was raised in the early days of kinetic theory, both by Maxwell and Boltzmann. In the real world, most phenomena involving many-particle systems do take place in a geometry driven by boundaries or external fields. But the Boltzmann and Vlasov equations are still poorly understood in this respect. Even for the hypoelliptic kinetic Fokker-Planck operators in a domain, there is nothing equivalent to the huge body of works on the eigenvalue problem of the Laplace equation in a domain. The majority of the results and challenges discussed before lend themselves to boundary-driven formulations, which are mostly open.

Specific boundary-related problems raise beautiful challenges: propagation of singularities according to the shape, ergodicity and relaxation to equilibrium…

An even more ambitious goal is the understanding of *self-induced nontrivial geometry*, as observed in particular in galactic dynamics, where the geometry of the confinement is influenced by the gravitational mean-field of the system itself.

9. Surveys

In this last section we list some survey books and synthesis articles for further reading. We start with a **very short selection** of books and survey articles which can be used by a reader who wants to dwelve into the subject.

- E. M. Lifshitz & L. P. Pitaevski. *Teoreticheskaya fizika* (”Landau–Lifshitz”), Tom 10. (Russian) Fizicheskaya kinetika. [Physical kinetics] “Nauka”, Moscow, 1979.
- N. A. Krall & A. W. Trivelpiece. *Principles of Plasma Physics.* San Francisco Press, 1986.
- J. Binney & S. Tremaine. *Galactic dynamics.* Princeton University Press, 1987.
- C. Cercignani. *Rarefied gas dynamics – From basic concepts to actual calculations.* Cambridge Texts in Applied Mathematics, Cambridge University Press, 2000.
- C. Cercignani, R. Illner and M. Pulvirenti. *The mathematical theory of dilute gases.* Springer, New York, 1994.
- P. A. Markowich, C. A. Ringhofer & C. Schmeiser. *Semiconductor equations.* Springer-Verlag, Vienna, 1990.
- R. Glassey. *The Cauchy problem in kinetic theory.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1996.
- C. Villani. *A review of mathematical topics in collisional kinetic theory*, Handbook of mathematical fluid dynamics, Vol. I, 71-305, North-Holland, Amsterdam, 2002.
- C. Villani. *Irreversibility & Entropy* (French & English versions), Proceedings of the 2010 Poincaré seminar on “Time”. Progress in Mathematical Physics, Birkhaüser, 2013.
- C. Villani. Mathematics of granular materials. *J. Stat. Phys. 124*, 2-4 (2006), 781–822.

In the rest of this section, we provide a longer list of articles and surveys (including the ones mentioned just above) which can be useful to the reader. We shall separate them into several categories.

For a start, here are some of the **founding works** for kinetic theory:

- J. C. Maxwell. *On the dynamical theory of gases.* J. Philosophical Transactions of the Royal Society of London, Vol. 157 (1867), 49–88.
- L. Boltzmann. *Weitere Studien über das Wärmegleichgewicht unter Gasmolekülen.* Wiener Berichte, Sitzungsberichte der Akademie der Wissenschaften 66 (1872), 275–370.
- L. Boltzmann. *Vorlesungen ueber Gastheorie.* Leipzig, J. A. Barth, 2 volumes, 1896 & 1898.
- H. Lorentz. Le mouvement des électrons dans les métaux. *Arch. Néerl. 10* (1905), 336–371.

- J. H. Jeans. *On the theory of star-streaming and the structure of the universe.* Monthly Notices of the Royal Astronomical Society 76 (1915), 70–84.
- A. A. Vlasov. *On vibrational properties of an electron gas* (in Russian). J. Exp. Theor. Phys. 8 (1938), No.3, 291–318.

The **history of the subject** is described in many references, such as

- S. Brush. *Kinetic Theory of Gases: An Anthology of Classic Papers With Historical Commentary*; History of Modern Physical Sciences, Imperial College Press, 1965. (This book includes translations of some of he above-mentioned founding works, edited and commented.)
- C. Cercignani. *Ludwig Boltzmann, The man who trusted atoms.* Oxford University Press, 1998.
- D. Lindley. *Boltzmann's Atom: The great debate that launched a revolution in physics.* Free Press, 2001.
- M. Hénon. *Vlasov Equation?* Astron. Astrophys. 114 (1982), 211–212. (A short historical comment on the birth of the Vlasov equation.)
- C.D. Levermore. *History of Kinetic Theory.* `http://www.terpconnect.umd.edu/~lvrmr/History/index.shtml` (Unfinished project of a Webpage on kinetic theory, containing incomplete but still relevant historical information.)

For **general presentations** and treatises of kinetic theory, from the point of view of physics and modelling, one can consult:

- E. M. Lifshitz & L. P. Pitaevski. *Teoreticheskaya fizika* ("Landau–Lifshitz"), Tom 10. (Russian) Fizicheskaya kinetika. [Physical kinetics] “Nauka”, Moscow, 1979.
- S. Chapman & T. G. Cowling. *The mathematical theory of non-uniform gases – An account of the kinetic theory of viscosity, thermal conduction and diffusion in gases.* Third edition, prepared in co-operation with D. Burnett, Cambridge University Press, London, 1970.
- C. Cercignani. *Theory and application of the Boltzmann equation.* Elsevier, New York, 1975.
- R. Balescu. *Equilibrium and nonequilibrium statistical mechanics.* Wiley-Interscience, New York–London–Sydney, 1975.
- A. Akhiezer, I. Akhiezer, R. Polovin, A. Sitenko & K. Stepanov. *Plasma electrodynamics. Vol. I: Linear theory, Vol. II: Non-linear theory and fluctuations.* Translated into English by D. ter Haar. Pergamon Press, 1975.
- N. A. Krall & A. W. Trivelpiece. *Principles of Plasma Physics.* San Francisco Press, 1986.

- J. Binney & S. Tremaine. *Galactic dynamics.* Princeton University Press, 1987.
- H. Risken. *The Fokker–Planck equation, Methods of solution and applications.* Springer-Verlag, 1989.
- J.L. Delcroix and A. Bers. *Physique des Plasmas* (in French), 2 volumes. Intereditions/CNRS, 1994.
- J. Lebowitz. Microscopic reversibility and macroscopic behavior: physical explanatoins *[sic]* and mathematical derivations. In *Twenty-five years of non-equilibrium statistical mechanics, proceedings of the XIII Sitges conference* (1994), J. Brey, J. Marro, J. Rubi, and M. S. Miguel, Eds., Lect. Notes in Physics, Springer, pp. 1–20.
- C. Cercignani, R. Illner and M. Pulvirenti. *The mathematical theory of dilute gases.* Springer, New York, 1994.
- C. Cercignani. *Recent developments in the mechanics of granular materials.* In *Fisica matematica e ingeneria delle strutture : rapporti e compatibilità*, G. Ferrarese, Ed. Pitagora Ed., Bologna, 1995, pp. 119–132.
- A. Decoster. Survey of the collisional kinetic theory of plasmas, Part I in *Modeling of collisions*, by A. Decoster, P.A. Markowich and B. Perthame. Gauthier-Villars, Elsevier, Paris, 1998.
- D.D. Ryutov. Landau damping: half a century with the great discovery. *Plasma Phys. Control. Fusion 41* (1999), A1–A12.
- C. Cercignani. *Rarefied gas dynamics – From basic concepts to actual calculations.* Cambridge Texts in Applied Mathematics, Cambridge University Press, 2000.

Here are some **more mathematically oriented surveys**, varying in size, timing and scope:

- T. Carleman. *Problèmes mathématiques dans la théorie cinétique des gaz* (completed by L. Carleson). Publ. Sci. Inst. Mittag-Leffler, Vol. 2. Almqvist & Wiksells Boktryckeri Ab, Uppsala 1957.
- A.W. Weinberg & E.P. Wigner. *The physical theory of neutron chain reactors.* The University of Chicago Press, 1958.
- P. Ehrenfest & T. Ehrenfest. *The conceptual foundations of the statistical approach in mechanics.* Dover Publications Inc., New York, 1990. Translated from the 1959 German original by Michael J. Moravcsik. With a foreword by M. Kac and G.E. Uhlenbeck.
- M. Kac. *Probability and related topics in physical sciences.* Interscience Publishers, London–New York, 1959.

- G.E. Uhlenbeck & G.W. Ford. *Lectures in Statistical Mechanics.* Chap. 3: The Boltzmann equation.
- K.M. Case & P.F. Zweifel. *Linear transport theory.* Addislon-Wesley, 1967.
- R.L. Drake. *A general mathematical survey of the coagulation equation.* Topics in Current Aerosol Research, 1972.
- C. Truesdell & R. G. Muncaster. *Fundamentals of Maxwell's kinetic theory of a simple monatomic gas – Treated as a branch of rational mechanics.* Pure and Applied Mathematics, Vol. 83. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York–London, 1980.
- R. Dautray & J.L. Lions. *Mathematical Analysis and Numerical Methods for Science and Technology.* Vol. 6: Evolution Problems II. Chapter XXI: *Transport* (pp. 209–416). Contributions by C. Bardos, M. Cessenat, A. Kavenoky, B. Mercier and R. Sentis. Translated from the 1985 French Edition by A. Craig. Springer, 2000.
- R. Illner & H. Neunzert. On simulation methods for the Boltzmann equation. *Transport Theor. Stat. Phys. 16* (1987), 141–154.
- A. V. Bobylev. *The theory of the nonlinear spatially uniform Boltzmann equation for Maxwell molecules.* In *Mathematical Physics Reviews, 7*, Vol. 7 of *Soviet. Sci. Rev. Sect. C Math. Phys. Rev. Harwood Academic Publ. Chur*, 1988, pp. 111–233.
- P. A. Markowich, C. A. Ringhofer & C. Schmeiser. *Semiconductor equations.* Springer-Verlag, Vienna, 1990.
- H. Spohn. *Large Scale Dynamics of Interacting Particles.* Texts and Monographs in Physics, Springer-Verlag, Heidelberg, 1991.
- G.A. Bird. *Molecular Gas Dynamics and the direct simulation of gas flows.* Oxford University Press, Second Ed., 1994.
- R. Glassey. *The Cauchy problem in kinetic theory.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1996.
- C. Villani. *Entropy production and convergence to equilibrium.* Expanded version of notes from a series of lectures at the Institut Henri Poincaré, Paris (Fall 2001). In *Entropy methods for the Boltzmann equation*, *Lecture Notes in Math.*, Vol. 1916, Springer, Berlin, 2008, pp. 1–70.
- Y. Sone. *Kinetic theory and fluid dynamics.* Modeling and Simulation in Science, Engineering and Technology. Birkhäuser Boston, Inc., Boston, 2002.
- C. Villani. *A review of mathematical topics in collisional kinetic theory*, Handbook of mathematical fluid dynamics, Vol. I, 71-305, North-Holland, Amsterdam, 2002.

- Ph. Laurençot & S. Mischler. On coalescence equations and related models. In *Modeling and computational methods for kinetic equations*, Model. Simul. Sci. Eng. Technol. Birkhaüser Boston, 2004, pp. 321–356.
- H. Andréasson. *The Einstein-Vlasov System/Kinetic Theory.* Living Rev. Relativity, Vol. 8, 2005.
- C. Villani. Mathematics of granular materials. *J. Stat. Phys. 124*, 2-4 (2006), 781–822.
- S.-H. Yu. The development of the Green's function for the Boltzmann equation. *J. Stat. Phys. 124*, 2-4 (2006), 301–320.
- C. Villani. H-Theorem and Beyond: Boltzmann's entropy in today's mathematics. Proceedings of the Boltzmann memorial conferences in Vienna and Munich (July & October 2006). In *Boltzmann's legacy, ESI Lect. Math. Phys., Eur. Math. Soc., Zürich*, 2008, pp. 129–143.
- C. Villani. Hypocoercive diffusion operators. In *International Congress of Mathematicians, Madrid, 2006.* Vol. III, Eur. Math. Soc., pp.473–498.
- C. Mouhot. Quantitative linearized study of the Boltzmann collision operator and applications. *Commun. Math. Sci. Suppl. 1* (2007), 73–86.
- Y. Sone. *Molecular Gas Dynamics.* Birkhaüser, 2007.
- A. Naranyan & A. Klöckner. *Deterministic numerical schemes for the Boltzmann equation.* Brown University Technical Report, 2007. Arxiv: 0911.3589.
- A. D. Rendall. *Partial differential equations in general relativity.* Vol. 16 of *Oxford Graduate Texts in Mathematics.* Oxford University Press, 2008.
- L. Saint-Raymond. *Hydrodynamic limits of the Boltzmann equation.* Lectures at SISSA, Trieste. Lecture Notes in Mathematics, Springer, 2008.
- R. Alexandre. A review of Boltzmann equation with singular kernels. *Kinet. Relat. Models 2*, 4 (2009), 551–646.
- T.-P. Liu & S.-H. Yu. Boltzmann equation, boundary effects. *Discrete Contin. Dyn. Syst. 24*, 1 (2009), 145–157.
- C. Villani. *Hypocoercivity.* Memoirs of the American Mathematical Society, 2009.
- L. Desvillettes, C. Mouhot & C. Villani. Celebrating Cercignani's conjecture for the Boltzmann equation. *Kinet. Relat. Models 4*, 1 (2011), 277–294.
- I. Gallagher, L. Saint-Raymond & B. Texier. *From Newton to Boltzmann: Hard spheres and short-range potentials.* To appear in *Zürich Advanced Lectures in Mathematics.*
- C. Villani. *Landau Damping.* To appear in *Numerical Models of Fusion*, Panoramas & Synthèses, Société Mathématique de France, Ed.

- C. Villani. *Irreversibility & Entropy* (French & English versions), Proceedings of the 2010 Poincaré seminar on "Time". Progress in Mathematical Physics, Birkhaüser, 2013.
- C. Villani. Particle systems and nonlinear Landau damping. Proceedings of the meeting of the Division of Plasma Physics of the American Physics Society, Salt Lake City, Nov. 2011. To appear in *Physics of Plasmas*.

Finally, we further mention the **Porto Ercole lecture notes** — This is a series of high level lecture notes from the biennial summer schools *Methods and Models of Kinetic Theory*, held in Porto Ercole; they were published in the *Rivista di Matematica della Università di Parma*, starting from 2002.

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