Existence and Regularity of Solutions to the Minimal Surface Equation

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1 Introduction

Minimal surfaces date back to the introduction of calculus of variations and were studied by both Euler and Lagrange; indeed we will shortly present a derivation of the minimal surface equation as the Euler-Lagrange equations of the area functional on graphs. In this report we will be concerned with establishing existence and regularity properties of solutions to the minimal surface equation. We proceed by constructing first a Lipschitz continuous solution, before showing that this solution necessarily satisfies additional regularity properties. There are many approaches to the latter problem, the one we present very PDE theoretic and is based, in part, on the celebrated theory of DeGiorgi, Nash and Moser. Finally we will drop the Lipschitz condition on our boundary data and generalise to arbitrary continuous boundary conditions. As promised though, we first derive the minimal surface equation by way of motivation.

1.1 Derivation of the Minimal Surface Equation

Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain (that is, $\Omega$ is open and connected). Fix $\phi : \partial \Omega \to \mathbb{R}$, and introduce

$$\mathcal{L}(\Omega; \phi) := \{ u \in C^{0,1}(\Omega); \, u|_{\partial \Omega} = \phi \},$$

the set of Lipschitz functions on $\Omega$ whose restriction to $\partial \Omega$ is $\phi$. We consider the problem of minimising the area functional

$$\mathcal{A}(u) := \int_{\Omega} \sqrt{1 + |Du|^2}$$

over functions $u \in \mathcal{L}(\Omega; \phi)$. Note that since Lipschitz functions are almost everywhere differentiable, we can make sense of this functional for all $u \in \mathcal{L}(\Omega; \phi)$. Suppose $u \in \mathcal{L}(\Omega; \phi)$ is a minimiser and that $v \in C^\infty_c(\Omega)$. Clearly for all $t \in \mathbb{R}$, $u + tv \in \mathcal{L}(\Omega; \phi)$. Therefore the function

$$f : t \mapsto \mathcal{A}(u + tv)$$

has a local minimum at $t = 0$, and so $f'(0) = 0$. Differentiating $f$ and evaluating at $t = 0$ leads to

$$f'(0) = \int_{\Omega} \frac{Du \cdot Dv}{\sqrt{1 + |Du|^2}} = 0,$$

(1.3)
which is precisely what it means for \( u \) to be a weak solution of the equation

\[
\text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = 0.
\]

Using the summation convention we may rewrite this in the following equivalent form

\[
\mathcal{M}u := D_i \left( \frac{D_i u}{\sqrt{1 + |Du|^2}} \right) = 0 \quad \text{in } \Omega.
\] (1.4)

This is known as the minimal surface equation in divergence form and will be the object of our study for the remainder of this report.

**Remark.** Returning to (1.3), we note that if we assume \( u \in C^2(\Omega) \), then integrating by parts yields

\[
0 = \int_{\Omega} \frac{Du \cdot Dv}{\sqrt{1 + |Du|^2}} = - \int_{\Omega} \text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) v,
\]

with the boundary terms vanishing since \( v \) is compactly supported in \( \Omega \). Since \( v \in C_c^\infty(\Omega) \) was arbitrary, we conclude that \( u \) satisfies (1.4) in the classical sense as we would expect.

We now observe that equation (1.4) may be expanded as follows:

\[
\mathcal{M}u = D_i \left( \frac{D_i u}{\sqrt{1 + |Du|^2}} \right) = \left[ \frac{D_{ii} u}{\sqrt{1 + |Du|^2}} - \frac{D_i u D_{ij} u D_j u}{(1 + |Du|^2)^{3/2}} \right].
\]

which may be equivalently expressed as

\[
\sum_{i,j=1}^{n} \left( \delta_{ij} - \frac{D_i u D_j u}{1 + |Du|^2} \right) D_{ij} u = 0.
\] (1.5)

From equation (1.5) we immediately recognize that this is a second order quasilinear PDE, since the coefficients of the highest order terms depend only on lower order terms. A quick calculation now establishes that the minimal surface equation is elliptic. First we let

\[
a_{ij}(x, Du) = \delta_{ij} - \frac{D_i u D_j u}{1 + |Du|^2}.
\]

Recall that the PDE (1.5) is elliptic if for each \( x \) and \( u \) there exists a \( \lambda > 0 \) such that \( a_{ij}(x, Du) \xi_i \xi_j \geq \lambda |\xi|^2 \) for all \( \xi \in \mathbb{R}^n \). We observe

\[
a_{ij}(x, Du) \xi_i \xi_j = \left( \delta_{ij} - \frac{D_i u D_j u}{1 + |Du|^2} \right) \xi_i \xi_j
\]

\[
= |\xi|^2 - \frac{D_i u \xi_i D_j u \xi_j}{1 + |Du|^2}
\]

Now by the Cauchy-Schwarz inequality we have \((Du \cdot \xi)^2 \leq |Du|^2 |\xi|^2\) which implies

\[
a_{ij}(x, Du) \xi_i \xi_j \geq |\xi|^2 \left( 1 - \frac{|Du|^2}{1 + |Du|^2} \right).
\] (1.6)

Since \(|Du|^2/(1 + |Du|^2) < 1\), (1.6) implies \( a_{ij}(x, Du) \xi_i \xi_j \geq \lambda |\xi|^2 \) which establishes that the PDE in (1.5) is elliptic.
2 Existence of Solutions

In this section we prove the existence of Lipschitz continuous solutions \( u \) to the minimal surface equation (1.3), with given boundary data \( \phi \). We assume throughout that our domain \( \Omega \subset \mathbb{R}^n \) is bounded, with \( C^2 \) boundary. For now we assume that \( \phi \in C(\partial \Omega) \), later on we will need to make additional assumptions on the regularity of \( \phi \) to deduce existence. Most of the material in this section is based on [3] and [5], with many details expanded upon.

2.1 The functional \( A \) and its properties

We first prove a few basic properties of the area functional \( A \), as defined in (1.2). Note that we can make sense of \( A(u) \) for any \( u \in W^{1,1}(\Omega) \). The following property will be useful in various proofs throughout this section:

**Theorem 2.1.** For any function \( u \in W^{1,1}(\Omega) \),

\[
\int_{\Omega} \sqrt{1 + |Du|^2} \, dx = \sup \left\{ \int_{\Omega} (g_{n+1} + u \, \text{div} \, g) \, dx ; \ g \in C^1_c(\Omega; \mathbb{R}^{n+1}), \ |g|_\infty \leq 1 \right\}.
\]

**Proof.** We first show that

\[
\int_{\Omega} |Dv| \, dx = \sup \left\{ \int_{\Omega} (v \, \text{div} \, g) \, dx ; \ g \in C^1_c(\Omega'; \mathbb{R}^n), \ |g|_\infty \leq 1 \right\}
\]

for all \( v \in W^{1,1}(\Omega') \) where \( \Omega' \subset \mathbb{R}^{n+1} \). To see this we use Cauchy Schwartz and integration by parts to get

\[
\int_{\Omega} (v \, \text{div} \, g) \, dx = \int_{\Omega} (Dv) \cdot (-g) \, dx \leq \int_{\Omega} |Dv| |g| \, dx \leq \int_{\Omega} |Dv| \, dx,
\]

and the other direction follows on choosing \( g \) to be approximations to \( -Dv/|Dv| \) in \( C^1_c \). To establish (2.1) let \( u \in C^1(\Omega) \) and introduce \( v := -x_{n+1} + u \). Integrating by parts we have

\[
\int_{\Omega} (g_{n+1} + u \, \text{div} \, g) \, dx = \int_{\Omega} (g_{n+1} - \sum_{i=1}^{n+1} \frac{\partial u}{\partial x_i} g_i) \, dx = \int_{\Omega} (g_{n+1} - \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} g_i) \, dx = \int_{\Omega} (v \, \text{div} \, g) \, dx,
\]

Noting that \( |Dv| = \sqrt{1 + |Du|^2} \) and the result follows from (2.2).

**Corollary 2.2** (Weak Lower Semicontinuity). Let \( \{u_j\}_{j \in \mathbb{N}} \subset W^{1,1}(\Omega) \) be a sequence of functions which converge in \( L^1(\Omega) \) to a function \( u \in W^{1,1}(\Omega) \). Then

\[ A(u) \leq \liminf_{j \to \infty} A(u_j). \]

**Proof.** Let \( g \in C^1_c(\Omega; \mathbb{R}^{n+1}) \), with \( |g|_\infty \leq 1 \). Then we have

\[
\int_{\Omega} (g_{n+1} + u \, \text{div} \, g) \, dx = \lim_{j \to \infty} \int_{\Omega} (g_{n+1} + u_j \, \text{div} \, g) \, dx \leq \liminf_{j \to \infty} A(u_j)
\]

The first equality holds since \( g \) is compactly supported and continuously differentiable and so \( \text{div} \, g \) is bounded on \( \Omega \). Weak lower semi-continuity now follows on taking the supremum over all \( g \).
Corollary 2.3 (Strict Convexity). The area functional $A$ is strictly convex in the following sense: if $u, v \in W^{1,1}(\Omega)$ are such that $Du \neq Dv$ then for all $t \in (0, 1)$

$$A(tu + (1-t)v) < tA(u) + (1-t)A(v)$$

Proof. This follows immediately from the strict convexity of the function $x \mapsto \sqrt{1 + |x|^2}$, which follows from the fact that the Hessian is positive definite at every $x$. □

We can make sense of the area functional for any $u \in W^{1,1}(\Omega)$, and in fact, any $u \in BV(\Omega)$. For our purposes however it will be enough to restrict our attention to Lipschitz continuous functions on $\Omega$, which are necessarily $W^{1,1}(\Omega)$ and so the area functional remains well defined on this class of functions.

Theorem 2.4 (Characterisation of $W^{1,\infty}$). Let $\Omega$ be open and bounded, with $\partial\Omega$ of class $C^2$. Then $u : \Omega \to \mathbb{R}$ is Lipschitz continuous if and only if $u \in W^{1,\infty}(\Omega)$.

2.2 Lipschitz Functions and Minimisers of $A$

The next step is to choose a suitable space over which to minimise $A$. We want to choose a suitably large space in order that we may appeal to certain compactness properties in order to show existence, while at the same time not discarding so much regularity that proving additional regularity later becomes too hard. It turns out that the space of Lipschitz continuous functions strikes the right balance. We first present some definitions and useful facts. For a function $u$ which is Lipschitz continuous on $\Omega$, we denote its Lipschitz constant as

$$[u]_\Omega = \sup \left\{ \frac{|u(x) - u(y)|}{|x-y|} : x, y \in \Omega, x \neq y \right\}.$$ 

Definition. Let $k > 0$, and set $\mathcal{L}_k(\Omega)$ to be the space of Lipschitz continuous functions with Lipschitz constant less than or equal to $k$, i.e.

$$\mathcal{L}_k(\Omega) = \{ u \in C^{0,1}(\Omega) : [u]_\Omega \leq k \}.$$

For boundary data $\phi \in C(\partial\Omega)$, let $\mathcal{L}(\Omega; \phi)$ be the set of Lipschitz continuous functions which agree with $\phi$ on $\partial\Omega$, as defined in (1.1). Finally, let $\mathcal{L}_k(\Omega; \phi)$ be the set of Lipschitz continuous functions with Lipschitz constant less than or equal to $k$, and which agree with $\phi$ on $\partial\Omega$, i.e.

$$\mathcal{L}_k(\Omega; \phi) = \mathcal{L}_k(\Omega) \cap \mathcal{L}(\Omega; \phi).$$

Now we present two results about sequences of functions in these spaces. Combining them with the Arzelà-Ascoli theorem will allow us to conclude compactness of the space $\mathcal{L}_k(\Omega; \phi)$.

Lemma 2.5. Any sequence of functions $\{u_j\} \subset \mathcal{L}_k(\Omega)$ is equicontinuous in $\Omega$.

Proof. Pick $\varepsilon > 0$, and let $\delta < \frac{\varepsilon}{k}$. Then for any $x, y \in \Omega$ with $|x-y| < \delta$, we have

$$|u_j(x) - u_j(y)| \leq k|x-y| < k\delta < \varepsilon,$$

for any $j \in \mathbb{N}$. Hence the sequence is equicontinuous. □
Lemma 2.6. Any sequence of functions \( \{u_j\} \subset \mathcal{L}_k(\Omega; \phi) \) is uniformly bounded in \( \Omega \).

Proof. Let \( d_{\text{max}} := \sup \{ |x - y| : x, y \in \Omega \} \) be the greatest distance between any two points in \( \Omega \). Since we are assuming that \( \Omega \) is bounded, we know that \( d_{\text{max}} \) is finite. By compactness of \( \partial \Omega \) and continuity of \( \phi \), we also know that \( \|\phi\|_{L^\infty(\partial \Omega)} \) is finite. Therefore, using the uniform bound of \( k \) on the Lipschitz constants, we can see that

\[
|u_j(x)| \leq \|\phi\|_{L^\infty(\partial \Omega)} + kd_{\text{max}} < \infty
\]

for any \( x \in \Omega \) and \( j \in \mathbb{N} \). Hence the sequence is uniformly bounded. \( \square \)

Using the above results, we can now very easily deduce the existence of a minimiser in the class \( \mathcal{L}_k(\Omega, \phi) \). From here it will take some extra work, as well as some additional assumptions on the domain \( \Omega \), to show the existence of a minimiser in \( \mathcal{L}(\Omega, \phi) \).

Theorem 2.7. Let \( \phi \in C(\partial \Omega) \), and suppose that \( \mathcal{L}_k(\Omega, \phi) \) is non-empty. Then there exists a function \( u[k] \in \mathcal{L}_k(\Omega, \phi) \) such that

\[
\mathcal{A}(u[k]) = \inf \{ \mathcal{A}(u) : u \in \mathcal{L}_k(\Omega, \phi) \}.
\]

Proof. Let \( \{u_j\}_{j \in \mathbb{N}} \) be a minimising sequence in \( \mathcal{L}_k(\Omega, \phi) \). By Lemmas 2.5 and 2.6 the sequence is equicontinuous and uniformly bounded. We can therefore apply the Arzelà-Ascoli theorem to find a subsequence which converges uniformly to a function \( u[k] \in \mathcal{L}(\Omega, \phi) \). Since convergence is uniform and \( \Omega \) is bounded it follows that \( u_j \) converge to \( u[k] \) in \( L^1 \). Hence by weak lower semicontinuity of \( \mathcal{A} \) we deduce that

\[
\mathcal{A}(u[k]) \leq \lim \inf_{j \to \infty} \mathcal{A}(u_j) = \inf \{ \mathcal{A}(u) : u \in \mathcal{L}_k(\Omega, \phi) \} \leq \mathcal{A}(u[k]),
\]

which completes the proof. \( \square \)

Proposition 2.8. Suppose \( u[k] \in \mathcal{L}_k(\Omega, \phi) \) is the minimum defined above for some \( k \), that is

\[
\mathcal{A}(u[k]) = \inf \{ \mathcal{A}(v) : v \in \mathcal{L}_k(\Omega, \phi) \}.
\]

If in addition we have the strict inequality \( [u[k]]_\Omega < k \), then \( u[k] \) is the minimum of \( \mathcal{A} \) in \( \mathcal{L}(\Omega, \phi) \), so

\[
\mathcal{A}(u[k]) = \inf \{ \mathcal{A}(v) : v \in \mathcal{L}(\Omega, \phi) \}.
\]

Proof. Let \( 0 \leq t \leq 1 \) and \( v \in \mathcal{L}(\Omega, \phi) \), and set \( v_t = u[k] + t(v - u[k]) \). Then note that \( v_t|_{\partial \Omega} = \phi \) and if \( t \) is sufficiently small then \( [v_t]_\Omega < k \). To see this, find \( \varepsilon > 0 \) such that \( [u[k]]_\Omega = k - \varepsilon \) and choose \( N \) such that \( |v - u[k]|_\Omega \leq N \) (which is possible, since both \( v \) and \( u[k] \) Lipschitz). Then take \( t = \varepsilon/2N \) and use the triangle inequality. Since \( u[k] \) is the minimiser of \( \mathcal{A} \) in \( \mathcal{L}_k(\Omega, \phi) \), we must then have \( \mathcal{A}(u[k]) \leq \mathcal{A}(v_t) \). Thus by the convexity of the area functional we have

\[
\mathcal{A}(u[k]) \leq \mathcal{A}(v_t) \leq (1 - t)\mathcal{A}(u[k]) + t\mathcal{A}(v),
\]

which, upon rearrangement, completes the proof. \( \square \)
To finish the proof of existence it remains to show two things. First, we must show that for \( k \) large enough, \( L_k(\Omega; \phi) \) is non-empty. This is trivial since we have assumed a \( C^2 \) domain with continuous boundary data. Hence we can produce a Lipschitz function agreeing with \( \phi \) on the boundary simply by solving the Dirichlet problem for the Laplacian. According to Proposition 2.8 to show the existence of a minimiser in \( L(\Omega, \phi) \) it is sufficient to obtain estimates on the Lipschitz constant of the minimisers \( u^k \) in \( L_k(\Omega, \phi) \). We do this by formulating a weak maximum principle which will allow us to restrict our attention to boundary estimates on the Lipschitz constant, and then obtain the desired estimates using the notion of a barrier function.

### 2.3 Maximum principle

**Definition** (Supersolutions and Subsolutions). Function \( w \in L_k(\Omega) \) is called a supersolution (subsolution) for the area functional \( A \) if for all \( v \in L_k(\Omega; w) \) with \( v \geq w (v \leq w) \) on \( \Omega \) we have \( A(v) \geq A(w) \).

**Proposition 2.9** (Weak maximum principle). Let \( w \) and \( z \) be a supersolution and a subsolution of \( A \) in \( L_k(\Omega) \) respectively. If \( w \geq z \) on the boundary \( \partial \Omega \) then \( w \geq z \) in \( \overline{\Omega} \).

**Proof.** Seeking a contradiction suppose that the maximum principle doesn’t hold, then the set \( K := \{ x \in \Omega : w(x) < z(x) \} \) is non empty. Let \( v := \max \{ z, w \} \). Then it is clear that \( v = w \) on the boundary, \( v \in L_k(\Omega) \) and \( v \geq w \) in \( \Omega \). Thus, since \( w \) is a supersolution, we have \( A(v) \geq A(w) \). On the set \( K \), \( v = z \) and therefore the area functional on \( K \) satisfies \( A(z; K) \geq A(w; K) \). By considering \( \min \{ z, w \} \) we similarly see that \( A(z; K) \leq A(w; K) \), so we deduce

\[
A(z; K) = A(w; K).
\]

Since \( z = w \) on the boundary \( \partial K \) (as \( z \) and \( w \) are continuous), and \( z \geq w \) in \( K \), there exists a set of non zero measure where \( Dw \neq Dz \). By the strict convexity of the area functional we therefore get

\[
A\left(\frac{w + z}{2}; K\right) < \frac{1}{2}A(w; K) + \frac{1}{2}A(z; K) = A(w; K).
\]

But since \( w \) is a supersolution in \( L_k(\Omega) \) and \( (w + z)/2 \geq w \) on \( K \) we also have

\[
A\left(\frac{w + z}{2}; K\right) \geq A(w; K),
\]

giving the desired contradiction. \( \square \)

**Proposition 2.10.** Let \( w \) and \( z \) be a supersolution and a subsolution of \( A \) in \( L_k(\Omega) \) respectively. Then we have

\[
\sup_{x \in \Omega} [z(x) - w(x)] \leq \sup_{y \in \partial \Omega} [z(y) - w(y)]. \tag{2.3}
\]

**Proof.** First, we note that if \( w \) is a supersolution then so is \( w + \alpha \) for any \( \alpha \in \mathbb{R} \). To see this suppose \( v \in L_k(\Omega; w + \alpha) \) and \( v \geq w + \alpha \). Then \( v - \alpha \in L_k(\Omega; w) \) and \( v - \alpha \geq w \) so we get \( A(v - \alpha) \geq A(w) \). \( w + \alpha \) is then a supersolution since \( A(v - \alpha) = A(v) \). Now choose \( \alpha = \sup_{y \in \partial \Omega} [z(y) - w(y)] \), and note that for any \( x \in \partial \Omega \),
\[ z(x) \leq w(x) + \sup_{y \in \partial \Omega} [z(y) - w(y)]. \]

After applying the maximum principle we get that the above equation holds for all \( x \in \Omega \), and taking the supremum over all \( x \) gives (2.3).

**Corollary 2.11.** If \( u \) and \( v \) both minimise the area functional \( A \) in \( L_k(\Omega) \) then

\[ \sup_{x \in \Omega} |u(x) - v(x)| \leq \sup_{x \in \partial \Omega} |u(x) - v(x)|. \]

**Proof.** This comes immediately from Proposition 2.10, since any area minimising function is both a supersolution and a subsolution.

**Proposition 2.12** (Reduction to boundary estimates). Suppose \( u \) minimises the area functional in \( L_k(\Omega) \), then we have

\[ [u]_{\Omega} \leq \sup_{x \in \Omega, y \in \partial \Omega} \frac{|u(x) - u(y)|}{|x - y|}. \]

**Proof.** Let \( x_1, x_2 \) be two distinct points in \( \Omega \) and set \( r = x_2 - x_1 \). If \( u \) minimises area in \( L_k(\Omega) \) then \( u_r(x) := u(x + r) \) minimises area in \( L_k(\Omega_r) \), where \( \Omega_r := \{ x + r : x \in \Omega \} \). Note that \( x_2 \in \Omega \cap \Omega_r \) so the intersection is non empty, and that both \( u \) and \( u_r \) minimise area in \( L_k(\Omega \cap \Omega_r) \). We can apply Corollary 2.11, which gives us that there is some \( z \in \partial(\Omega \cap \Omega_r) \) such that

\[ |u(x_1) - u(x_2)| = |u_r(x_2) - u(x_2)| \leq |u_r(z) - u(z)| = |u(z + r) - u(z)|. \]

Then dividing by \( |r| = |x_1 - x_2| \) we get

\[ \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|} \leq \frac{|u(z) - u(z + r)|}{|r|}. \]

Since \( z \in \partial(\Omega \cap \Omega_r) \) then either \( z \in \partial \Omega \), or \( z \in \partial \Omega_r \) in which case \( z + r \in \partial \Omega \). Thus one of \( z \) or \( z + r \) lie on the boundary \( \partial \Omega \), so

\[ \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|} \leq \frac{|u(z) - u(z + r)|}{|r|} \leq \sup_{x \in \Omega, y \in \partial \Omega} \frac{|u(x) - u(y)|}{|x - y|}, \]

and the result follows by taking the supremum on the left hand side.

**2.4 Barriers and completion of proof of existence**

To complete the proof, we introduce the notion of a barrier, for which it is necessary to assume that \( \phi \) is Lipschitz. We show that the existence of barriers is sufficient for existence of a solution and then construct barriers under certain assumptions on \( \Omega \) to prove that a solution exists.

**Definition** (Upper and lower barriers). Let \( x \in \Omega \), and let \( d(x) = \text{dist}(x, \partial \Omega) \). For \( c > 0 \) define

\[ \Sigma_c := \{ x \in \Omega : d(x) < c \} \quad \text{and} \quad \Gamma_c := \{ x \in \Omega : d(x) = c \}. \]

Let \( \phi \in C^{0,1}(\partial \Omega) \). An upper barrier \( v^+ \) relative to \( \phi \) is a Lipschitz function defined on \( \overline{\Sigma_{c_o}} \) for some \( c_o \), which satisfies
(i) \( v^+|_{\partial \Omega} = \phi \) (\( v^+ \) agrees with \( \phi \) on the outer edge),

(ii) \( v^+ \geq \sup_{\partial \Omega} \phi \) on \( \Gamma_{c_0} \) (\( v^+ \) lies above \( \phi \) on the inner edge),

(iii) \( v^+ \) is a supersolution on \( \Sigma_{c_0} \).

A lower barrier \( v^- \) is a Lipschitz function if it satisfies (1), \( v^- \leq \inf_{\partial \Omega} \phi \) on \( \Gamma_{c_0} \) and is a subsolution on \( \Sigma_{c_0} \).

**Theorem 2.13.** Let \( \phi \) be a Lipschitz function on \( \partial \Omega \), and suppose upper and lower barriers \( v^+ \) and \( v^- \) (relative to \( \phi \)) exist. Then the area functional \( A \) achieves its minimum on \( L(\Omega; \phi) \).

**Proof.** Let \( Q := \max ([v^+]_{\Sigma_{c_0}} [v^-]_{\Sigma_{c_0}}) \), and choose \( k > Q \) large enough that \( L_k(\Omega, \phi) \) is non-empty. Following earlier arguments, there exists an area minimising function \( u \in L_k(\Omega, \phi) \). We claim that in fact \( |u|_\Omega < k \) and thus via Proposition 2.8 that \( u \) is the desired minimiser of \( A \) on \( L(\Omega, \phi) \). We begin by observing that \( u \) also minimises area in \( L_k(\Sigma_{c_0}) \). Moreover it follows that

\[
\inf_{\partial \Omega} \phi \leq u(x) \leq \sup_{\partial \Omega} \phi \quad \forall x \in \Omega
\]

To see this suppose the set \( K := \{ x \in \Omega : \sup_{\partial \Omega} \phi < u(x) \} \) is non-empty. Since \( u \) is continuous, it follows that \( K \) is open and furthermore since \( u \leq \sup_{\partial \Omega} \phi \) on \( \partial \Omega \), we have \( u = \sup_{\partial \Omega} \phi \) on \( \partial K \).

Now if \( u \) minimises \( A \) on \( K \), which in turn implies that \( u \equiv \sup_{\partial \Omega} \phi \) on \( K \), which is a contradiction. In particular we have \( v^-(x) \leq u(x) \leq v^+(x) \) for \( x \in \Gamma_{c_0} \), and thus by the weak maximum principle, we deduce

\[
v^-(x) \leq u(x) \leq v^+(x)
\]

for \( x \in \Sigma_{c_0} \). Since \( v^- = u = v^+ \) on \( \partial \Omega \), it now follows that if \( x \in \Sigma_{c_0} \) and \( y \in \partial \Omega \), then \( u(x) - u(y) \leq v^+(x) - v^+(y) \) if \( u(x) \geq u(y) \) and \( u(y) - u(x) \leq v^-(y) - v^-(x) \) if \( u(x) \leq u(y) \).

Therefore by the definition of \( Q \)

\[
|u(x) - u(y)| \leq Q|x - y|
\]

for \( x \in \Sigma_{c_0} \) and \( y \in \partial \Omega \).

We now extend this to arbitrary \( x \in \Omega \). Suppose \( d(x) > c_0 \), since \( \inf_{\partial \Omega} \phi \leq u(x) \leq \sup_{\partial \Omega} \phi \) we have the following inequality for \( y \in \partial \Omega \)

\[
|u(x) - u(y)| \leq \max \left\{ \sup_{\partial \Omega} \phi - u(y), u(y) - \inf_{\partial \Omega} \phi \right\} \leq \max \left\{ v^+(z) - v^+(y), v^-(y) - v^-(z) \right\}
\]

for any \( z \in \Gamma_{c_0} \). Choosing \( z \) with \( |z - y| = c_0 \) we conclude that

\[
|u(x) - u(y)| \leq \max \left\{ [v^+]_{\Sigma_{c_0}} c_0, [v^-]_{\Sigma_{c_0}} c_0 \right\} \leq Q|x - y|.
\]

In other words, by Proposition 2.12 \( |u|_\Omega \leq Q < k \) and therefore Proposition 2.8 implies that \( u \) is a minimiser in \( L(\Omega, \phi) \). \( \square \)

To complete the proof of existence, we will construct an upper barrier. This is sufficient since if \( v \) is an upper barrier relative to \( -\phi \) then \( -v \) is a lower barrier relative to \( \phi \). In order to do so we need to make additional assumptions about the domain \( \Omega \), and in particular we must introduce the notion of mean curvature.
Definition. Assume that $\partial \Omega$ is $C^2$. For $y \in \partial \Omega$ we denote by $\nu(y)$ and $T(y)$ respectively the inner normal to $\partial \Omega$ at $y$ and the tangent hyperplane to $\partial \Omega$ at $y$. The curvatures at $y$ are determined as follows. By a rotation of the coordinates we may assume that the $x_n$ coordinate axis lies in the direction $\nu(y)$. Then in some neighbourhood $N = N(y)$ of $y$, $\partial \Omega$ is then given by $x_n = \psi(x')$ where $x' = (x_1, \ldots, x_{n-1})$, $\psi \in C^2(T(y) \cap N)$ and $D\psi(y') = 0$. The eigenvalues $\kappa_1, \ldots, \kappa_{n-1}$ of the Hessian matrix $(D^2\psi(y'))$ are called the principal curvatures of $\partial \Omega$ at $y$, and the mean curvature of $\partial \Omega$ at $y$ is given by

$$H(y) = \frac{1}{n-1} \sum_{i=1}^{n-1} \kappa_i$$

We use the following facts about the distance function $d = \text{dist}(x, \partial \Omega)$, the proofs of which can be found in [2].

(i) For sufficiently small $c_0$, $d \in C^2(\Sigma_{c_0})$.

(ii) If $d(x) = c \leq c_0$ then $-\Delta d(x) = nH(x)$ where $H(x)$ is the mean curvature of $\Gamma_c$ at $x$.

(iii) If the mean curvature $H$ of $\partial \Omega$ is non-negative at every point, then $d$ is superharmonic in $\Sigma_{c_0}$ (so $\Delta d \leq 0$).

The assumption of non-negative mean curvature, also referred to as mean convexity, combined with these facts will allow us to construct an upper barrier, under the additional assumption that $\phi \in C^2(\partial \Omega)$. Later, in Section 5 we will show that the regularity assumptions on $\phi$ can be relaxed.

Theorem 2.14 (Existence of a Lipschitz solution to the MSE). Let $\Omega$ be a bounded $C^2$ domain whose boundary has non negative mean curvature, and $\phi \in C^2(\partial \Omega)$. Then there exists $u \in W^{1,\infty}(\Omega)$ which minimises the area functional $A$, i.e. $u$ solves

$$\int_{\Omega} \frac{Du \cdot D\eta}{\sqrt{1 + |Du|^2}} = 0$$

for all $\eta \in C^1_c(\Omega)$ and the boundary condition $u|_{\partial \Omega} = \phi$.

Proof. We will construct a barrier of the form

$$v(x) = \phi(x) + \psi(d(x)),$$

for some $\psi \in C^2[0, R]$ to be determined, where $R < c_0$, and $c_0$ is as above. We choose $\psi$ such that $\psi(0) = 0$ and

$$\psi(R) \geq 2 \sup_{\Omega} |\phi| =: L$$

so that $v$ satisfies the first and second properties of an upper barrier. We also specify that $\psi'(t) \geq 1$ and $\psi''(t) < 0$ which we will make use of shortly in our calculations. We just need to show that $v$ is a supersolution, and it is sufficient to show that $v$ satisfies

$$\mathcal{M} v = \sum_{i=1}^{n} D_i \left( \frac{D_i v}{\sqrt{1 + |Du|^2}} \right) \leq 0$$
for all \( x \in \Omega \), which is equivalent to
\[
\zeta(v) := (1 + |Dv|^2)^{3/2} Mv = (1 + |Dv|^2) \Delta v - \sum_{i,j=1}^n D_i v D_j v D_{ij} v \leq 0.
\]

A tedious calculation which the reader is encouraged to check yields (where we have used summation convention)
\[
\zeta(v) = (1 + |D\phi|^2) \Delta \phi - D_i \phi D_j \phi D_{ij} \phi \\
+ \psi' [2 D_i \phi D_j d \Delta \phi + (1 + |D\phi|^2) \Delta d - D_i d D_j \phi D_{ij} \phi - D_i \phi D_j \phi D_{ij} d] \\
+ \psi'' [\Delta \phi + 2 D_i \phi D_j d \Delta d - D_i D_j d D_{ij} \phi] + \psi^3 \Delta d \\
+ \psi'' [1 + |D\phi|^2 - (D_i \phi D_j d)^2]. \tag{2.4}
\]

This follows on using the property that \( |Dd| = 1 \), so \( D_i d D_{ij} d = 0 \), and also noting that \( D_i \psi(d) = \psi'' d D_{ij} d \) and \( D_{ij} \psi(d) = \psi'' D_i d D_j d + \psi' D_{ij} d \). We now use the fact that \( \psi' \geq 1 \), so we have \( \psi^2 \geq \psi' \).

We also note that \( \phi \) and \( d \) are \( C^2 \) in \( \Sigma_R \) and hence the second order derivatives are bounded. Thus we can bound the first three terms in (2.4) (up to the term involving \( \psi^3 \)) by \( C \psi^2 \) for some \( C \), dependent on the region and \( \phi \). We now turn our attention to the term involving \( \psi'' \), noting that \( |D\phi|^2 - (D_i \phi D_j d)^2 \geq |D\phi|^2 - |D\phi|^2 |Dd|^2 \geq 0 \) by Cauchy-Schwartz, and since \( \psi'' < 0 \) we then get then final term is no greater than \( \psi'' \). As a result, we reduce (2.4) to the inequality
\[
\zeta(v) \leq \psi'' + C \psi^2 + \psi^3 \Delta d,
\]
which further reduces to
\[
\zeta(v) \leq \psi'' + C \psi^2 \tag{2.5}
\]
since \( \Delta d \leq 0 \) in \( \Sigma_R \). It remains to choose \( \psi \) such that the right hand side of (2.5) is less than zero, and that \( \psi \) satisfies the technical conditions above. We can choose \( \psi \) by solving the differential equation \( \psi'' + C \psi^2 = 0 \) which gives
\[
\psi(t) = \frac{1}{C} \log(1 + \beta t) + \gamma
\]
We set \( \gamma = 0 \) so that \( \psi(0) = 0 \) and it is clear that \( \psi'' < 0 \), so it remains to choose \( \beta \) and \( R \) such that \( \psi \) is an upper barrier, i.e. such that \( \psi' \geq 1 \) and \( \psi(R) \geq L \). We calculate that
\[
\psi'(t) = \frac{1}{C} \frac{\beta}{1 + \beta t} \geq \frac{1}{C} \frac{\beta}{1 + \beta R}
\]
and
\[
\psi(R) = \frac{1}{C} \log(1 + \beta R),
\]
thus we need \( \beta/(1 + \beta R) \geq C \) and \( \beta R \geq e^{LC} - 1 \). One way of doing this is setting \( R = \beta^{-\frac{1}{2}} \), which gives us the inequalities \( \beta/(1 + \beta^{\frac{1}{2}}) \geq C \) and \( \beta^{\frac{1}{2}} \geq e^{LC} - 1 \). By taking \( \beta \) sufficiently large (and hence \( R \) sufficiently small) we construct an upper barrier, which completes the proof of existence of a Lipschitz solution by Theorem 2.13.
Note that the only part in the proof of existence that required \( \partial \Omega \) to be \( C^2 \) was in constructing an upper barrier. It’s a natural question to ask whether we can still be guaranteed a solution under weaker assumptions on the boundary. As we will show in the next section this is not the case. If only a single point on the boundary has negative mean curvature then it is impossible to construct boundary data for which there is no minimiser. The situation for \( \phi \) is better, and we will show later how we can reduce the regularity assumptions on \( \phi \) while still maintaining existence of a minimiser.

3  Necessity of mean convexity

In this section we show that mean convexity is essential in the proof of the existence theorem in Section 2. We will show that if there is a point where the mean curvature \( H \) is negative then we can find some \( \phi \in C^2(\partial \Omega) \) such that there is no solution to the minimal surface equation with these boundary data. To do this, we first present a version of the maximum principle.

Lemma 3.1. Let \( \Omega \) be a bounded \( C^2 \) domain and \( \Gamma \) a non-empty, closed subset of \( \partial \Omega \). Let \( u, v \in C^2(\Omega) \cap C(\Omega \cup \Gamma) \) satisfy

\[
\mathcal{M}v \leq \mathcal{M}u \text{ in } \Omega \quad \text{and} \quad v \geq u \text{ on } \Gamma.
\]

Let \( \Omega_t := \{ x \in \Omega : \text{dist}(x, \partial \Omega) \geq t \} \) and let \( \nu_t \) be the outward normal of \( \Omega_t \). If for every open set \( V \supset \Gamma \) we have

\[
\lim_{t \to 0^+} \int_{\partial \Omega_t \setminus V} \left( 1 - \frac{Dv(y) \cdot \nu_t(y)}{\sqrt{1 + |Dv(y)|^2}} \right) \mathcal{H}^{n-1} = 0,
\]

then we have

\[
v \geq u \text{ in } \Omega \cup \Gamma.
\]

Proof. Let \( \varphi = \max\{0, u - v\} \) and \( \varphi_k = \min\{k, \varphi\} \), we want to show that \( \varphi \equiv 0 \) in \( \Omega \). Since \( \mathcal{M}v \leq \mathcal{M}u \) in \( \Omega \) the same is true in \( \Omega_t \) for all \( t \), since \( \varphi_k \) is positive we have

\[
\int_{\Omega_t} (\mathcal{M}v - \mathcal{M}u) \varphi_k \leq 0. \tag{3.2}
\]

Define

\[
G(p) = \frac{p}{\sqrt{1 + |p|^2}} \text{ and } G'(p) = \frac{p}{\sqrt{1 + |p|^2}}.
\]

Note that \( \mathcal{M}w = D_t G'(Dw) \), so using integration by parts on (3.2) we get

\[
\int_{\partial \Omega_t} (G(Dv) - G(Du)) \cdot \nu_t \varphi_k - \int_{\Omega_t} (G(Dv) - G(Du)) \cdot D\varphi_k \leq 0
\]

Assume first \( v > u \) on \( \Gamma \), and set \( V = \{ x \in \Omega : u(x) < v(x) \} \). Notice we have \( \varphi_k = D\varphi_k \equiv 0 \) on \( V \) and that \( V \) is open (as a subset of \( \Omega \)) and contains \( \Gamma \). Thus we have

\[
\int_{\Omega_t \setminus V} (G(Du) - G(Dv)) \cdot D\varphi_k \leq \int_{\partial \Omega_t \setminus V} (G(Du) - G(Dv)) \cdot \nu_t \varphi_k. \tag{3.3}
\]
Now using the fundamental theorem of the calculus and the chain rule we see

\[ G^i(Du) - G^i(Dv) = \int_0^1 \frac{d}{dt} G^i(tDu + (1-t)Dv) \, dt \]
\[ = \int_0^1 \frac{\partial G^i}{\partial p_j} (tDu + (1-t)Dv) \frac{d}{dt} (tD_j u + (1-t)D_j v) \, dt \]
\[ = D_j(u-v) \int_0^1 \frac{\partial G^i}{\partial p_j} (tDu + (1-t)Dv) \, dt = D_j(u-v)b_{ij} \quad (3.4) \]

where we defined

\[ b_{ij} := \int_0^1 \frac{\partial G^i}{\partial p_j} (tDu + (1-t)Dv) \, dt. \]

Observe \( b_{ij} \) is elliptic. This follows from Section 1 since \( a_{ij} \) is elliptic due to \( Du \) and \( Dv \) being uniformly bounded. Thus we have \( b_{ij} \xi_i \xi_j \geq \lambda_i|\xi|^2 \) for some \( \lambda_i > 0 \) for \( x \in \Omega_t \). Proceeding, we have

\[
\int_{\Omega_t} \lambda_i |D\varphi_k|^2 \leq \int_{\Omega_t} b_{ij} D_j \varphi_k D_i \varphi_k \quad \text{by ellipticity}
\]
\[
= \int_{\Omega_t} b_{ij} D_j(u-v) D_i \varphi_k \quad \text{since } D_i \varphi_k = D(u-v) \text{ if } 0 < u-v < k \text{ and } 0 \text{ otherwise}
\]
\[
= \int_{\Omega_t \setminus \Gamma} b_{ij} D_j(u-v) D_i \varphi_k \quad \text{by definition of } V
\]
\[
\leq \int_{\partial \Omega_t \setminus \Gamma} (G(Du) - G(Dv)) \cdot \nu_i \varphi_k \quad \text{by (3.3) and (3.4)}
\]
\[
= \int_{\partial \Omega_t \setminus \Gamma} (1 - G(Dv) \cdot \nu_i) \varphi_k - \int_{\partial \Omega_t \setminus \Gamma} (1 - G(Du) \cdot \nu_i) \varphi_k. \quad (3.5)
\]

Since \( |G(Du) \cdot \nu_i| \leq 1 \) by the Cauchy-Schwartz inequality, \( \int_{\partial \Omega_t \setminus \Gamma} (1 - G(Dv) \cdot \nu_i) \varphi_k \geq 0 \) and thus (3.5) becomes

\[
\int_{\Omega_t} \lambda_i |D\varphi_k|^2 \leq \int_{\partial \Omega_t \setminus \Gamma} (1 - G(Dv) \cdot \nu_i) \varphi_k \leq k \int_{\partial \Omega_t \setminus \Gamma} (1 - G(Dv) \cdot \nu_i), \quad (3.6)
\]

where the second inequality follows from the definition of \( \varphi_k \). We now let \( t \searrow 0 \). By hypothesis the limit of the right hand side is zero so \( \lim_{t \searrow 0} \int_{\Omega_t} \lambda_i |D\varphi_k|^2 = 0 \), from which we can conclude \( |D\varphi_k| = 0 \). Since (3.6) holds for all values of \( k \), we can take \( k \to \infty \) and conclude that \( D\varphi \equiv 0 \) in \( \Omega \). This means that either \( u \leq v \) in \( \Omega \), or \( u-v \) is constant in \( \Omega \). If it is the second case then since \( u \leq v \) on \( \Gamma \) we must have by continuity that \( u \leq v \) in \( \Omega \). If we only have \( v \geq u \) on \( \Gamma \), we replace \( v \) by \( v + \varepsilon \) in the definition of \( V \) and then let \( \varepsilon \to 0 \) to get the result. \( \square \)

We now restate what we are trying to show, the non-existence of a solution.

**Theorem 3.2.** Let \( \Omega \) be a bounded \( C^2 \) domain and suppose that there is some \( x_0 \) in \( \partial \Omega \) with mean curvature \( H(x_0) < 0 \). Then there exists \( \phi \in C^2(\partial \Omega) \) such that the Dirichlet problem

\[
M u = 0 \text{ in } \Omega
\]
\[
u|_{\partial \Omega} = \phi
\]

has no solution \( u \in C^2(\overline{\Omega}) \).
Using the version of the maximum principle we have just proved, we now show the theorem is true. We prove an inequality related to the values that \( u \) can take on the boundary and then show we can easily construct boundary data for which this does not hold.

**Proof.** We assume that \( u \in C^2(\Omega) \) is a solution to the Minimal Surface Equation in \( \Omega \). Since \( H(x_0) < 0 \) then by properties of the distance function we stated in §2.4 we have \( \Delta d(x_0) \geq 0 \) where \( d(x) = \text{dist}(x, \partial \Omega) \) as before. Since \( d \in C^2(\Sigma_c) \) for sufficiently small \( c \) then there exists some \( R > 0 \) such that \( \Delta d(x) > \varepsilon > 0 \) for \( x \in \Omega \cap B_R(x_0) \). We claim that

\[
\sup_{\partial \Omega \cap B_R(x_0)} u < \sup_{\partial \Omega \setminus B_R(x_0)} u + C,
\]

where \( C \) depends only on the domain \( \Omega \). To prove this we apply Lemma 3.1 to two different functions. Let \( v(x) := \alpha - \beta d(x)^{1/2} \) for \( x \in \Omega \cap B_R(x_0) \) and let \( w(x) := \lambda - \mu \text{dist}(x, \partial B_R(x_0))^{1/2} \) for \( x \in \Omega \setminus B_R(x_0) \), with constants to be chosen later. We firstly deal with \( v(x) \), we can calculate that

\[
\mathcal{M} v = -\frac{\beta}{2\sqrt{d(x)} + \beta^2/4} \left( \Delta d(x) - \frac{1}{2(d(x) + \beta^2/4)} \right),
\]

by checking that \( D_i v = -\frac{1}{2} \beta D_i d d^{-1/2} \) and \( D_{ij} v = -\frac{1}{2} \beta D_{ij} d d^{-1/2} + \frac{1}{4} \beta D_i D_j d d^{-3/2} \) and recalling that \( |Dd| = 1 \) and \( D_i d D_{ij} d = 0 \).

Since \( \mathcal{M} u = 0 \) to apply the lemma we need that \( \mathcal{M} v \leq 0 \) in \( \Omega \cap B_R(x_0) \), and having \( \beta^2 > \frac{2}{\varepsilon} \) is sufficient since

\[
\Delta d(x) - \frac{1}{2(d(x) + \beta^2/4)} > \varepsilon - \frac{2}{\beta^2} > 0.
\]

**Figure 1:** Diagram of \( \Omega \) and \( B_R(x_0) \)

We want to use the Lemma in \( \Omega_1 := \Omega \cap B_R(x_0) \) and \( \Gamma_1 = \overline{\Omega} \cap \partial B_R(x_0) \). We check that (3.1) holds, to see this note that we have \( \nu_1(y) = -Dd(y) \) and so the integrand reduces to

\[
1 - \frac{Dv(y) \cdot \nu_1(y)}{\sqrt{1 + |Dv(y)|^2}} = 1 + \frac{\beta Dd(y) \cdot \nu_1(y)}{2\sqrt{d(y) + \beta^2/4}} = 1 - \frac{\beta}{2\sqrt{d(y) + \beta^2/4}}.
\]
which is zero on the boundary of $\Omega$. To apply the Lemma we just need to choose $\alpha > 0$ such that $v(x) \geq u(x)$ on $\Gamma_1$, and since $\alpha \geq v(x) \geq \alpha - \beta \sqrt{\text{diam} (\Omega)}$ we can choose

$$\alpha = \sup_{\Gamma_1} u + \beta \sqrt{\text{diam} (\Omega)}.$$ 

Applying the Lemma we then get for $x \in \Omega_1 \cup \Gamma_1$ that

$$u(x) \leq v(x) \leq \sup_{\Gamma_1} u + \beta \sqrt{\text{diam} (\Omega)}.$$ 

Thus we get upon taking a supremum in $\Omega_1$ that

$$\sup_{\Omega_1} u \leq \sup_{\text{p} \in \partial B(x_0)} u + \beta \sqrt{\text{diam} (\Omega)}.$$ (3.7)

Finally we can use the maximum principle (see [6]) which gives us $\sup_{\Omega_1} u = \sup_{\partial \Omega_1} u$ and thus since $\partial \Omega \cap B_R(x_0) \subset \partial \Omega_1$ from (3.7) we have

$$\sup_{\partial \Omega \cap B_R(x_0)} u \leq \sup_{\text{p} \in \partial B(x_0)} u + \beta \sqrt{\text{diam} (\Omega)}.$$ (3.8)

We now carry out a similar procedure for $w(x)$ with $x \in \Omega_2 := \Omega \setminus B_R(x_0)$. We can see that $w(x) = \lambda - \mu(|x - x_0| - R)^{1/2}$. We again need that $\mathcal{M}v \leq 0$ in $\Omega_2$ and a similar calculation as before (with $d(x) = |x - x_0| - R$) gives

$$\mathcal{M}v = -\frac{\mu}{2\sqrt{|x - x_0| - R + \mu^2/4}} \left( \frac{n-1}{|x - x_0|} - \frac{1}{2(|x - x_0| - R + \mu^2/4)} \right).$$

Since

$$\frac{n-1}{|x - x_0|} - \frac{1}{2(|x - x_0| - R + \mu^2/4)} \geq \frac{n-1}{|x - x_0|} - \frac{2}{\mu^2},$$

we can ensure the above expression is positive and hence $\mathcal{M}v \leq 0$ by taking

$$\mu > \sqrt{\frac{2 \text{diam} (\Omega)}{n-1}}.$$ 

By similar logic to before, we have $w(x) \geq \lambda - \mu \sqrt{\text{diam} (\Omega)}$ so choosing

$$\lambda = \sup_{\Gamma_2} u + \mu \sqrt{\text{diam} (\Omega)}$$

ensures that $v(x) \geq u(x)$ in $\Gamma_2 := \partial \Omega \setminus B_R(x_0)$. We can check that the integral in (3.1) is zero as before by showing the integrand is zero on the boundary, so we can apply the Lemma to get that

$$u(x) \leq w(x) \leq \sup_{\Gamma_1} u + \mu \sqrt{\text{diam} (\Omega)}.$$ 

Finally taking the supremum as before and applying the maximum principle in $\Omega_2$ we conclude that

$$\sup_{\text{p} \in \partial B(x_0)} u \leq \sup_{\partial \Omega \setminus B_R(x_0)} u + \beta \sqrt{\text{diam} (\Omega)}.$$ (3.9)

since $\overline{\Omega} \cap \partial B_R(x_0) \subset \partial \Omega_2$. Combining (3.8) and (3.9) gives us the result we wanted,

$$\sup_{\partial \Omega \cap B_R(x_0)} u \leq \sup_{\partial \Omega \setminus B_R(x_0)} u + C.$$ (3.10)
In fact we have just proved that we can find $\phi \in c^\infty(\Omega)$ for which there is no solution to the Dirichlet problem. For example choose $\phi \sim e^{-x^2}$ with peak at $2C$ in $\partial \Omega \cap B_R(x_0)$ and less than $C$ in $\partial \Omega \setminus B_R(x_0)$.

4 Regularity

Having established the existence of a Lipschitz continuous solution, which we will henceforth denote $\tilde{u}$, we now turn to the issue of regularity. Our goal is to show that the solution of the minimal surface equation that we have constructed is in fact smooth. This takes a lot of work, but the idea can be summarised roughly as follows. The first step is to prove that the solution $\tilde{u}$ is in fact in $W^{2,2}_{loc}$. Having done this we can rewrite the minimal surface equation to obtain a linear equation for the derivatives of $\tilde{u}$. We apply the theory of DeGiorgi-Nash-Moser to conclude that the derivatives are Hölder continuous. Since the coefficients in the minimal surface equation are given in terms of the derivatives of $\tilde{u}$, we may conclude that the coefficients are Hölder continuous. This will then allow us to show that $\tilde{u}$ is $C^{2,\alpha}$ and hence a classical solution of the minimal surface equation. We finally conclude by appealing to Schauder estimates and a bootstrapping argument to show that $\tilde{u}$ is $C^\infty$.

4.1 Additional Regularity of the Solution

We will use following results to show that the solution is in $W^{2,2}_{loc}$.

**Definition.** Assume that $u \in L^1_{loc}(\Omega)$, and $\Omega' \subset \subset \Omega$. Then the $i$-th difference quotient of size $h$, where $0 < |h| < \text{dist}(\Omega', \partial \Omega)$, is defined as

$$
\delta_i^h u(x) = \frac{u(x + he_i) - u(x)}{h},
$$

for $i = 1, \ldots, n$ and $x \in \Omega'$. We also define $\delta^h := (\delta_1^h, \ldots, \delta_n^h)$.

The first two theorems we will only state, proofs may be found in Evans [1].

**Theorem 4.1.** (i) Suppose that $1 \leq p < \infty$ and $u \in W^{1,p}(\Omega)$. Then for each $\Omega' \subset \subset \Omega$

$$
\|\delta^h u\|_{L^p(\Omega')} \leq C\|Du\|_{L^p(\Omega)}
$$

for some constant $C$ and for all $0 < |h| < \frac{1}{2}\text{dist}(\Omega', \partial \Omega)$.

(ii) Assume that $1 < p < \infty$, $u \in L^p(\Omega')$, and that there exists a constant $C$ such that

$$
\|\delta^h u\|_{L^p(\Omega')} \leq C,
$$

for all $0 < |h| < \frac{1}{2}\text{dist}(\Omega', \partial \Omega)$. Then

$$
u \in W^{1,p}(\Omega'), \quad \text{with } \|Du\|_{L^p(\Omega')} \leq C.
$$
Theorem 4.2. Let \( u \in W^{1,2}(\Omega) \cap W^{1,\infty}(\overline{\Omega}) \) satisfy
\[
\int_{\Omega} F_i(Du) D_i \varphi = 0, \tag{4.1}
\]
for all \( \varphi \in W^{1,2}_0(\Omega) \) with \( \varphi \geq 0 \), where \( F \) is a smooth function satisfying the following two conditions:

(i) \( D_i F_j = D_j F_i \) for all \( i, j \),

(ii) There exist \( c, C > 0 \) such that \( c|\xi|^2 \leq \sum_{i,j=1}^n D_j F_i \xi_i \xi_j \leq C|\xi|^2 \) for all \( \xi \in \mathbb{R}^n \), for all \( x \in \Omega \).

These conditions can be thought of as local uniform ellipticity conditions on \( F \), and under these assumptions we have \( u \in W^{2,2}_{\text{loc}} \).

Proof. We begin by constructing a suitable test function \( \varphi \in W^{1,2}_0(\Omega) \). In order to do so, let \( \eta \in C_\infty(\Omega) \) and \( |h| < \frac{1}{2}\text{dist}(\partial\Omega, \text{supp} \eta) \) and define \( \varphi := \delta_h \eta^2 \delta_k u \), where \( k \in \{1, \ldots, n\} \).

Also, note that \( \delta_k^h \) commutes with \( D_j \) and furthermore \( \delta_i^h \) commutes with \( D \) when \( \delta_i^h \) is applied componentwise. A quick calculation shows that by substituting our choice of test function into (4.1) we get
\[
\int_{\Omega} \delta_k^h F_i(Du) D_i (\eta^2 \delta_k^h u) dx = - \int_{\Omega} \delta_k^h F_i(Du) D_i (\eta^2 \delta_k^h u) dx,
\]
since \( \eta \) is compactly supported in \( \Omega \) and \( 0 < |h| < \frac{1}{2}\text{dist}(\partial\Omega, \text{supp} \eta) \). By (4.1), the LHS of the above expression is 0, so we have
\[
\int_{\Omega} \delta_k^h F_i(Du) D_i (\eta^2 \delta_k^h u) dx = 0. \tag{4.2}
\]

Now note that
\[
F_i(Du(x + he_k)) - F_i(Du(x)) = F_i(Du(x) + h\delta_k^h Du(x)) - F_i(Du(x)).
\]

Hence, by the fundamental theorem of calculus,
\[
\delta_k^h F_i(Du(x)) = \frac{1}{h} \int_0^1 \frac{d}{dt} F_i(Du(x) + th\delta_k^h Du(x)) dt
\]
\[
= D_j \delta_k^h u(x) \int_0^1 D_j F_i(Du(x) + th\delta_k^h Du(x)) dt.
\]

Now set
\[
\theta_{ij} := \int_0^1 D_j F_i(Du + th\delta_k^h Du) dt.
\]

Now if we set \( v := \delta_k^h u \), then we may rewrite (4.2) as
\[
\int_{\Omega} \theta_{ij} D_j v D_i (\eta^2 v) dx = 0. \tag{4.3}
\]
Suppose we are given \( \Omega_1 \subset \Omega \), then we can find sets \( \Omega_2 \) and \( \Omega_3 \) such that \( \Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \Omega \). In addition, choose \( |h| < \min_k \text{dist}(\partial \Omega_k, \Omega_{k+1}) \). Now choose \( \eta \in C_c(\Omega_2) \) such that \( \eta = 1 \) on \( \Omega_1 \). By the ellipticity assumption, we find that there exist constants \( c \) and \( C \) such that
\[
c|\xi|^2 \leq \theta_{ij} \xi_i \xi_j \leq C|\xi|^2
\]
in \( \Omega_2 \). Now since \( \eta = 1 \) on \( \Omega_1 \), we have
\[
\int_{\Omega_1} |Dv|^2 \leq \int_{\Omega_2} |D(\eta v)|^2.
\]
Using the first inequality in (4.4), we also have
\[
\int_{\Omega_2} |D(\eta v)|^2 \leq \frac{1}{c} \int_{\Omega_2} \theta_{ij} D_j(v \eta) D_i(v \eta),
\]
and moreover by the first ellipticity condition, we have \( \theta_{ij} = \theta_{ji} \) which implies
\[
\frac{1}{c} \int_{\Omega_2} \theta_{ij} D_j(v \eta) D_i(v \eta) = \frac{1}{c} \int_{\Omega_2} \theta_{ij} (v^2 D_j \eta D_i \eta + 2 \eta v D_j v D_i \eta + \eta^2 D_j v D_i v).
\]
Applying the product rule and the fact that \( \int_\Omega \theta_{ij} D_j v D_i (\eta^2 v) dx = 0 \) and (4.3) we find
\[
\frac{1}{c} \int_{\Omega_2} \theta_{ij} (v^2 D_j \eta D_i \eta + 2 \eta v D_j v D_i \eta + \eta^2 D_j v D_i v) = \frac{1}{c} \int_{\Omega_2} \theta_{ij} v^2 D_j \eta D_i \eta + \frac{1}{c} \int_{\Omega} \theta_{ij} D_j v D_i (\eta^2 v)
\]
\[
= \frac{1}{c} \int_{\Omega} \theta_{ij} v^2 D_j \eta D_i \eta.
\]
Finally using the second inequality in (4.4)
\[
\frac{1}{c} \int_\Omega \theta_{ij} v^2 D_j \eta D_i \eta \leq \frac{C}{c} \int_{\Omega_2} v^2 |D\eta|^2 \leq C \int_{\Omega_2} v^2,
\]
and therefore
\[
\int_{\Omega_1} |Dv|^2 \leq C \int_{\Omega_2} v^2.
\]
Now since \( v = \delta_k^h u \) and \( u \in W^{1,2}_{\text{loc}}(\Omega) \) we have that \( v \in L^2(\Omega_2) \) and
\[
||v||_{L^2(\Omega_2)} \leq C ||Du||_{L^2(\Omega_3)} =: K.
\]
By (4.5) and the above result, we have
\[
\int_{\Omega_1} |Dv|^2 \leq CK.
\]
Hence, \( ||\delta_k^h Du||_{L^2(\Omega_1)} \leq \tilde{K} \) which implies \( ||D^2 u||_{L^2(\Omega_1)} \leq \tilde{K} \) and therefore \( u \in W^{2,2}_{\text{loc}}(\Omega) \).

Applying the above results to our specific situation, we first note that \( \tilde{u} \) is Lipschitz and thus is \( W^{1,\infty}(\Omega) \). Hence since \( \Omega \) is bounded, we also have that \( \tilde{u} \in W^{1,2}(\Omega) \). We can therefore apply theorem 4.2 with \( F(p) := p/\sqrt{1+|p|^2} \) which satisfies the ellipticity conditions to conclude that \( \tilde{u} \in W^{2,2}_{\text{loc}} \).
4.2 DeGiorgi-Nash-Moser

4.2.1 Overview

The proof proceeds in multiple stages, many of which are very technical, so in the interest of clarity we present a brief informal overview of the strategy. We have just shown that $\tilde{u} \in W^{2,2}_{loc}$. With this result in hand we perform the following trick. Fix some $k = 1, \ldots, n$, let $w := D_k \tilde{u}$ and choose a test function of the form $D_k \phi$ for $\phi \in C_c^\infty(\Omega)$. Then substituting this into (1.3) we have

$$0 = \int_\Omega \frac{D_i \tilde{u} D_i(D_k \phi)}{\sqrt{1 + |D\tilde{u}|^2}} = -\int_\Omega \tilde{a}_{ij}(x) D_j w D_i \phi,$$

where

$$\tilde{a}_{ij}(x) = \frac{\delta_{ij}}{\sqrt{1 + |D\tilde{u}|^2}} - \frac{D_i \tilde{u} D_j \tilde{u}}{(1 + |D\tilde{u}|^2)^{3/2}}.$$

In other words, $w$ is a weak solution of the equation

$$D_i(\tilde{a}_{ij}(x) D_j u) = 0. \quad (4.7)$$

Moreover, this equation is linear! The coefficients depend only on $x$ since the solution $\tilde{u}$ is fixed. In addition to this, we also have ellipticity and boundedness of the coefficients $\tilde{a}_{ij}$. Observe

$$|\tilde{a}_{ij}(x)| = \left| \frac{\delta_{ij}}{\sqrt{1 + |D\tilde{u}|^2}} - \frac{D_i \tilde{u} D_j \tilde{u}}{(1 + |D\tilde{u}|^2)^{3/2}} \right| \leq \frac{1}{\sqrt{1 + |D\tilde{u}|^2}} \left( 1 + \frac{|D\tilde{u}|^2}{1 + |D\tilde{u}|^2} \right) \leq 2,$$

and

$$\tilde{a}_{ij} \xi_i \xi_j = \frac{1}{\sqrt{1 + |D\tilde{u}|^2}} \left( |\xi|^2 - \frac{D_i \tilde{u} \xi_i D_j \tilde{u} \xi_j}{1 + |D\tilde{u}|^2} \right) \geq \frac{|\xi|^2}{\sqrt{1 + |D\tilde{u}|^2}} \left( 1 - \frac{|D\tilde{u}|^2}{1 + |D\tilde{u}|^2} \right) \geq \lambda |\xi|^2,$$

where

$$\lambda = \inf_{\Omega} \frac{1}{(1 + |D\tilde{u}|^2)^{3/2}} > 0.$$

This last inequality follows since $\tilde{u} \in W^{1,\infty}(\Omega)$. We now study the function $w$ as a weak solution of the uniformly elliptic linear equation with bounded coefficients (4.7), and show that $w$ is Hölder continuous, which of course implies that $\tilde{u} \in C^{1,\alpha}$. This is done by appealing to the DeGiorgi-Nash-Moser theory. We first prove a local boundedness result using a powerful technique called Moser iteration. Specifically we show that for any ball $B_R \subset \subset \Omega$ and a subsolution $u \in W^{1,2}(B_R)$ we have that $u^+ \in L_{loc}^\infty(B_R)$ together with an estimate of the form

$$\sup_{B_r} u^+ \leq C \frac{1}{(R-r)^n} \|u^+\|_{L^p(B_R)}.$$

This will then allow us to prove a weak Harnack inequality of the form

$$\sup_{B_R} u \leq C \inf_{B_{R/2}} u.$$

Using this we can then show that the oscillation of $u$ (which we will define in due course) decays and this will allow us to deduce Hölder continuity. Finally it is an easy matter to apply this theory to the function $w$ and conclude the desired result.
4.2.2 Local Boundedness

From here on we consider a uniformly elliptic second order partial differential equation with bounded coefficients written in divergence form which takes the following form

\[ Lu = D_i (a_{ij}(x) D_j u) = 0. \] (4.8)

This is of course precisely the form of the equation \([4.7]\), however, we do not need to appeal to the specific structure of the minimal surface equation to prove the results which follow. We introduce the notions of weak subsolutions and supersolutions of \((4.8)\).

**Definition.** We say that \(u \in W^{1,2}(\Omega)\) is a subsolution (respectively supersolution) of \((4.8)\) if for any \(\phi \in W^{1,2}_0(\Omega)\) with \(\phi \geq 0\) we have

\[ \int_{\Omega} a_{ij} D_i u D_j \phi \leq 0 \quad \text{(resp.} \geq 0). \]

**Theorem 4.3** (Local Boundedness). Suppose that \(a_{ij} \in L^\infty(B_1)\) with \(\|a_{ij}\|_{L^\infty} \leq \Lambda\) and

\[ a_{ij}(x)\xi_i \xi_j \geq \lambda |\xi|^2 \quad \text{for all} \ x \in B_1, \xi \in \mathbb{R}^n, \]

for some \(\Lambda\) and \(\lambda\) positive. Suppose that \(u \in W^{1,2}(B_1)\) is a subsolution. Then \(u^+ \in L^\infty_{loc}(B_1)\).

Moreover, for each \(\theta \in (0, 1)\) and \(p > 0\) we have

\[ \sup_{B_\theta} u^+ \leq \frac{C}{(1 - \theta)^{\frac{1}{p}}} \|u^+\|_{L^p(B_1)}, \]

where \(C = C(n, \lambda, \Lambda, p)\) is a positive constant.

**Remark.** As mentioned in the overview, we will use a powerful technique known as Moser iteration to prove this. The idea is essentially as follows. By a clever choice of test function we estimate the \(L^{p_1}\) norm of \(u^+\) on a ball \(B_{r_1}\) by the \(L^{p_2}\) norm of \(u^+\) on a larger ball \(B_{r_2}\) where \(p_1 > p_2\), that is,

\[ \|u^+\|_{L^{p_1}(B_{r_1})} \leq C \|u^+\|_{L^{p_2}(B_{r_2})}. \]

We iterate this process for a careful choice of sequences \(r_i\) and \(p_i\) with \(r_i \searrow r > 0\) and \(p_i \to \infty\) to get the inequality we desire. The details now follow, but we will first state a technical lemma that will be needed towards the end of the proof of Theorem \(4.3\).

**Lemma 4.4.** Suppose that \(f(t)\) is bounded in \([\tau_0, \tau_1]\), with \(\tau_0 \geq 0\). Suppose that for all \(\tau_0 \leq t < s \leq \tau_1\) we have

\[ f(t) \leq \theta f(s) + \frac{A}{(s-t)^\alpha} \] (4.9)

for some fixed \(\theta \in [0, 1)\). Then for all \(\tau_0 \leq t < s \leq \tau_1\) we have

\[ f(t) \leq c(\alpha, \theta) \frac{A}{(s-t)^\alpha}. \]
Proof. Fix $\tau_0 \leq t < s \leq \tau_1$. Let $\tau \in (0, 1)$ and consider the sequence $(t_n)_{n=0}^{\infty}$ defined as follows:

\[
t_0 = t, \quad t_{n+1} = t_n + (1 - \tau)\tau^n(s - t).
\]

Notice that $\lim_{n \to \infty} t_n = s$ and that $(t_n)$ is an increasing sequence. Applying (4.9) repeatedly we get that

\[
f(t) = f(t_0) \leq \theta^nf(t_n) + \frac{A(s - t)^{-\alpha}}{(1 - \tau)^\alpha} \sum_{k=0}^{n-1} \theta^k(1 - \tau)^{-k\alpha}.
\]

Now choose $\tau$ such that $\theta \tau^{-\alpha} < 1$, then letting $n \to \infty$ we obtain

\[
f(t) \leq c(\alpha, \theta) \frac{A(s - t)^{-\alpha}}{(1 - \tau)^\alpha}.
\]

\[\square\]

Proof of Theorem 4.3. Step 1: We will first prove the special case $\theta = \frac{1}{2}$ and $p = 2$. Let $k, m > 0$ and define the functions $\overline{u} := u^+ + k$ and

\[
\overline{u}_m := \left\{ \begin{array}{ll} \overline{u} & \text{if } u < m \\ k + m & \text{if } u \geq m \end{array} \right.
\]

Note $D\overline{u}_m = 0$ if $u < 0$ or $u \geq m$ and furthermore $\overline{u}_m \leq \overline{u}$. Let $\beta \geq 0, \eta \in C^1_c(B_1)$ and introduce the test function

\[
\phi := \eta^2(\overline{u}_m^\beta \overline{u} - k^{\beta+1}) \in W^{1,2}_0(B_1).
\]

Since $\overline{u} \geq \overline{u}_m \geq k$ it is clear that $\phi \geq 0$. Computing $D\phi$ we see

\[
D\phi = \beta\eta^2 \overline{u}_m^{\beta-1} \overline{D\overline{u}_m} + \eta^2 \overline{u}_m^\beta \overline{D\overline{u}} + 2\eta D\eta(\overline{u}_m^\beta \overline{u} - k^{\beta+1})
\]

\[
= \eta^2 \overline{u}_m^\beta(\beta D\overline{u}_m + D\overline{u}) + 2\eta D\eta(\overline{u}_m^\beta \overline{u} - k^{\beta+1}),
\]

where we used the fact that $\overline{u}_m^\beta \overline{u} - k^{\beta+1}$.

Note that $\phi = 0$ and $D\phi = 0$ if $u \leq 0$. Moreover $\overline{u}$ is a subsolution since $u^+$ is and $D_j \overline{u} = D_i u^+$. Therefore we have

\[
0 \geq \int a_{ij}D_i \overline{u}D_j \phi = \int a_{ij}D_i \overline{u}(\beta D_j \overline{u}_m + D_j \overline{u})\eta^2 \overline{u}_m^\beta + 2 \int a_{ij}D_i \overline{u}(\overline{u}_m^\beta \overline{u} - k^{\beta+1}) D_j \eta.
\]

We have omitted the domain of integration for now, since these estimates hold when integrating over any ball $B_r(y) \subset B_1$. Using uniform ellipticity we also have the estimates

\[
\beta \int \eta^2 \overline{u}_m^\beta a_{ij}D_i \overline{u}D_j \overline{u}_m \geq \lambda \beta \int \eta^2 \overline{u}_m^\beta |D \overline{u}_m|^2, \quad \text{and}
\]

\[
\int \eta^2 \overline{u}_m^\beta a_{ij}D_i \overline{u}D_j \overline{u} \geq \lambda \int \eta^2 \overline{u}_m^\beta |D \overline{u}|^2.
\]

since either $D_i \overline{u} = D_i \overline{u}_m$ or $D_j \overline{u}_m = 0$ and $a_{ij}D_i \overline{u}_m D_j \overline{u}_m \geq \lambda |D \overline{u}_m|^2$. Furthermore, we may use Hölder’s inequality to obtain the following estimate for the second integral in (4.10), after first using the Cauchy-Schwarz inequality.

\[
2 \int a_{ij}D_i \overline{u}(\overline{u}_m^\beta \overline{u} - k^{\beta+1}) D_j \eta \eta \geq -2\lambda \int \eta \overline{u}_m^\beta \overline{u}|D \overline{u}||D\eta| \geq -2\lambda \left( \int |D \overline{u}|^2 \eta^2 \overline{u}_m^\beta \right)^{\frac{1}{2}} \left( \int |D\eta|^2 \overline{u}_m^\beta \right)^{\frac{1}{2}}.
\]

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Combining the last three inequalities with (4.10), we get

\[ 0 \geq \lambda \beta \int \eta^2 \overline{\omega}_m^\beta |D \overline{u}_m|^2 + \lambda \left( \int \eta^2 \overline{\omega}_m^\beta |D \overline{u}|^2 - \frac{2 \Lambda}{\lambda} \left( \int |D \overline{u}|^2 \eta^2 \overline{\omega}_m^\beta \right)^{\frac{1}{2}} \left( \int |D \eta|^2 \overline{\omega}_m^\beta \right)^{\frac{1}{2}} \right) \]
\[ \geq \lambda \beta \int \eta^2 \overline{\omega}_m^\beta |D \overline{u}_m|^2 + \lambda \int \eta^2 \overline{\omega}_m^\beta |D \overline{u}|^2 - \frac{2 \Lambda^2}{\lambda} \int |D \eta|^2 \overline{\omega}_m^\beta, \]

where we have used the inequality \( ab \leq \frac{a^2}{2} + \frac{b^2}{2} \) with \( a = \left( \int |D \overline{u}|^2 \eta^2 \overline{\omega}_m^\beta \right)^{\frac{1}{2}} \) and \( b = \frac{2 \Lambda}{\lambda} \left( \int |D \eta|^2 \overline{\omega}_m^\beta \right)^{\frac{1}{2}} \).

We deduce

\[ \beta \int \eta^2 \overline{\omega}_m^\beta |D \overline{u}_m|^2 + \int \eta^2 \overline{\omega}_m^\beta |D \overline{u}|^2 \leq 2 \left( \beta \int \eta^2 \overline{\omega}_m^\beta |D \overline{u}_m|^2 + \frac{1}{2} \int \eta^2 \overline{\omega}_m^\beta |D \overline{u}|^2 \right) \leq C \int |D \eta|^2 \overline{\omega}_m^\beta. \]

(4.14)

Set \( w := \beta \overline{u}_m \overline{u} \). Then note that

\[ |Dw|^2 = \left( \frac{\beta}{2} \overline{\omega}_m^{\beta-1} \overline{\omega}_m D \overline{u}_m + \overline{\omega}_m^\beta |D \overline{u}|^2 \right)^2 \]
\[ \leq \left( \frac{\beta}{2} \overline{\omega}_m^{\beta} |D \overline{u}_m| + |\overline{\omega}_m^\beta |D \overline{u}|^2 \right)^2 \]
\[ = |\overline{\omega}_m^\beta |D \overline{u}_m|^2 + \beta |\overline{\omega}_m^\beta |D \overline{u}_m|^2 + \frac{\beta^2}{4} |\overline{\omega}_m^\beta |D \overline{u}_m|^2 \]
\[ \leq (1 + \beta) (\beta \overline{\omega}_m^\beta |D \overline{u}_m|^2 + \overline{\omega}_m^\beta |D \overline{u}|^2). \]

The second line follows the fact that if \( \overline{u} \neq \overline{u}_m \) then \( D \overline{u}_m = 0 \). Then from (4.14) we have

\[ \int |Dw|^2 \eta^2 \leq C (1 + \beta) \int |D \eta|^2 \overline{\omega}_m^\beta = C (1 + \beta) \int |D \eta|^2 w^2, \]

which implies by the product rule that

\[ \int |D(w \eta)|^2 \leq C (1 + \beta) \int |D \eta|^2 w^2. \]

We apply the Gagliardo-Nirenberg-Sobolev inequality with \( \chi := \frac{n}{n^2} > 1 \) if \( n > 2 \) or \( \chi > 2 \) if \( n = 2 \), and obtain

\[ \left( \int |w \eta|^2 \right)^{\frac{\chi}{2}} \leq \int |D(w \eta)|^2 \leq C (1 + \beta) \int |D \eta|^2 w^2. \]

We will now choose a particular cut-off function \( \eta \). Let \( 0 < r < R \leq 1 \) and choose \( \eta \in C^1_c(B_R) \) with

\[ \eta \equiv 1 \text{ in } B_r \quad \text{and} \quad |D \eta| \leq \frac{2}{R - r}. \]

Then

\[ \left( \int_{B_r} w^{2\chi} \right)^{\frac{1}{\chi}} \leq C \frac{1 + \beta}{(R - r)^2} \int_{B_r} w^2. \]
Recalling how \( w \) was defined, this is the same as
\[
\left( \int_{B_r} w^2 \right)^{\frac{1}{2}} \leq C \frac{1 + \beta}{(R - r)^2} \int_{B_R} w^2.
\]
Set \( \gamma := \beta + 2 \geq 2 \) and recall that \( \overline{u}_m \leq \overline{u} \) so that
\[
\left( \int_{B_r} \overline{u}_m^{\gamma} \right)^{\frac{1}{\gamma}} \leq C \frac{\gamma - 1}{(R - r)^2} \int_{B_R} \overline{u}^\gamma.
\]
Let \( m \to \infty \) and conclude
\[
\| \overline{u} \|_{L^\gamma(B_r)} \leq \left( C \frac{\gamma - 1}{(R - r)^2} \right)^{\frac{1}{\gamma}} \| \overline{u} \|_{L^\gamma(B_R)},
\]
where \( C = C(n, \lambda, \Lambda) \) is independent of \( \gamma \).

**Step 2:** Define for \( i = 0, 1, \ldots \) the sequences \((\gamma_i)\) and \((r_i)\) by
\[
\gamma_i := 2^i, \quad r_i := \frac{1}{2} + \frac{1}{2^{i+1}}.
\]
Then we have \( r_i - r_{i-1} = 1/2^{i+1} \) and \( \gamma_i = \chi \gamma_{i-1} \), so
\[
\| \overline{u} \|_{L^{\gamma_i}(B_{r_i})} \leq \left( C \frac{2^{i-1}}{(1/2^{i+1})^2} \right)^{\frac{1}{\gamma_i}} \| \overline{u} \|_{L^{\gamma_{i-1}}(B_{r_{i-1}})}
\]
\[
\leq (32C) \frac{1}{2^{i+1}} \| \overline{u} \|_{L^{\gamma_{i-1}}(B_{r_{i-1}})}.
\]
Importantly, \( C \) depends only on \( n, \lambda \) and \( \Lambda \). Therefore we can iterate this \( l \) times to obtain
\[
\| \overline{u} \|_{L^{\gamma_l}(B_{r_l})} \leq (16C) \sum_{i=1}^{\gamma_l} \frac{1}{2^{i+1}} \| \overline{u} \|_{L^{\gamma_{i-1}}(B_{r_{i-1}})} \leq C \| \overline{u} \|_{L^2(B_{r_0})},
\]
where we have used the fact that the series \( \sum_{i=1}^{\infty} \frac{1}{2^i} \) and \( \sum_{i=1}^{\infty} \frac{i-1}{2^{i+1}} \) are both convergent since \( \chi > 1 \). In particular
\[
\| \overline{u} \|_{L^{\gamma_l}(B_{r_{1/2}})} \leq C \| \overline{u} \|_{L^2(B_1)}
\]
for all \( l \). Now let \( l \to \infty \). We have \( \gamma_l \to \infty \) also and therefore
\[
\sup_{B_{1/2}} \overline{u} \leq C \| \overline{u} \|_{L^2(B_1)}.
\]
Letting \( k \downarrow 0 \) in the definition of \( \overline{u} \) we arrive at the desired result
\[
\sup_{B_{1/2}} u^+ \leq C \| u^+ \|_{L^2(B_1)}.
\]

**Step 3:** We now seek to generalise for \( p > 0 \) and \( \theta \in (0, 1) \). We proceed by applying a dilation argument. Let \( R \leq 1 \) and define \( \hat{u}(y) := u(Ry) \) for \( y \in B_1 \). Then it is trivial to verify that
\[
\int_{B_1} \hat{u}_{ij} D_i \hat{u} D_j \phi \leq 0 \quad \forall \phi \in W^{1,2}_0(B_1), \text{ with } \phi \geq 0,
\]
where \( \hat{a}_{ij}(y) = a_{ij}(Ry) \). Notice that

\[
\|\hat{a}_{ij}\|_{L^\infty(B_1)} = \|a_{ij}\|_{L^\infty(B_R)} \leq \Lambda,
\]

and

\[
\hat{a}_{ij}(y)\xi_i \xi_j \geq \lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \forall y \in B_1.
\]

Hence we may apply the first part of the proof to \( \hat{u} \), and we write the resulting estimate in terms of \( u \). This yields, for \( p \geq 2 \) the following estimate

\[
\sup_{B_{R/2}} u^+ \leq \frac{C}{R^p} \|u^+\|_{L^p(B_R)},
\]

where \( C = C(n, \lambda, \Lambda, p) > 0 \). For \( \theta \in (0, 1) \), we apply the above result on \( B_{(1-\theta)R}(y) \) for each \( y \in B_{\theta R} \). This leads to

\[
u^+(y) \leq \sup_{B_{(1-\theta)R/2}(y)} u^+ \leq \frac{C}{((1-\theta)R)^p} \|u^+\|_{L^p(B_{(1-\theta)R}(y))} \leq \frac{C}{((1-\theta)R)^p} \|u^+\|_{L^p(B_R)},
\]

which implies

\[
\sup_{B_{R^\theta}} u^+ \leq \frac{C}{((1-\theta)R)^p} \|u^+\|_{L^p(B_R)}.
\]

For \( R = 1 \) this is precisely the stated result for \( p \geq 2 \).

**Step 4:** Our final goal is to extend this result to \( p \in (0, 2) \) also. Notice first that

\[
\int_{B_R} (u^+)^2 \leq \|u^+\|_{L^\infty(B_R)}^{2-p} \int_{B_R} (u^+)^p,
\]

and so

\[
\|u^+\|_{L^\infty(B_{\theta R})} \leq \frac{C}{((1-\theta)R)^p} \|u^+\|_{L^\infty(B_R)}^{1-\frac{p}{2}} \left(\int_{B_R} (u^+)^p\right)^{\frac{1}{2}}.
\]

We apply Young’s inequality

\[
ab \leq \frac{\varepsilon}{q} a^q + \frac{1}{\varepsilon^r} b^r \quad \text{for all} \quad a, b \geq 0 \quad \frac{1}{q} + \frac{1}{r} = 1 \quad \varepsilon > 0,
\]

where we choose

\[
a = \|u^+\|_{L^\infty(B_R)}^{1-\frac{p}{2}}, \quad b = \frac{C}{((1-\theta)R)^p} \left(\int_{B_R} (u^+)^p\right)^{\frac{1}{2}}, \quad q = \frac{2}{2-p}, \quad r = \frac{2}{p}, \quad \varepsilon = \frac{q}{2}.
\]

This yields

\[
\|u^+\|_{L^\infty(B_{\theta R})} \leq \frac{1}{2} \|u^+\|_{L^\infty(B_R)} + \frac{C}{((1-\theta)R)^p} \|u^+\|_{L^p(B_R)}.
\]

Setting \( f(t) := \|u^+\|_{L^\infty(B_t)} \) for each \( t \in (0, 1] \) we have shown that for any \( 0 < r < R \leq 1 \)

\[
f(r) \leq \frac{1}{2} f(R) + \frac{C}{(R-r)^p} \|u^+\|_{L^p(B_1)}.
\]
Therefore we may invoke Lemma \[4.4\] to deduce
\[ f(r) \leq \frac{C}{(R - r)^\frac{p}{n}} \|u^+\|_{L^p(B_1)}. \]

Let \( R \nearrow 1 \), then for all \( \theta < 1 \) we have
\[ \|u^+\|_{L^\infty(B_\theta)} \leq \frac{C}{(1 - \theta)^\frac{p}{n}} \|u^+\|_{L^p(B_1)} \]
as desired, thus we are done. \( \square \)

An immediate consequence of Theorem \[4.3\] is the following which we obtain by a scaling argument.

**Theorem 4.5.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) and
\[ Lu = D_i(a_{ij}(x)D_ju), \]
where \( \|a_{ij}\|_{L^\infty(\Omega)} \leq \Lambda \) and \( a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \) for all \( x \in \Omega, \xi \in \mathbb{R}^n \). Suppose that \( u \in W^{1,2}(\Omega) \) is a subsolution of \( L \) in \( \Omega \), that is
\[ \int_\Omega a_{ij}D_iuD_j\phi \leq 0 \]
for all \( \phi \in W^{1,2}_0(\Omega) \) with \( \phi \geq 0 \) in \( \Omega \). Then for any ball \( B_R(x) \subset \Omega, r < R \) and \( p > 0 \) we have
\[ \sup_{B_r} u^+ \leq \frac{C}{(R - r)^\frac{p}{n}} \|u^+\|_{L^p(B_R)}, \tag{4.15} \]
where \( C = C(n, \lambda, \Lambda, p) \) is positive.

### 4.2.3 Weak Harnack Inequality

**Theorem 4.6** (John-Nirenberg Lemma). Suppose \( u \in L^1(\Omega) \) satisfies
\[ \int_{B_r(x)} |u - u_{B_r(x)}| \leq Mr^n \quad \forall B_r(x) \subset \Omega, \]
where \( u_{B_r(x)} = \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)dy \). Then for any \( B_r(x) \subset \Omega \)
\[ \int_{B_r(x)} e^{p_0|u - u_{B_r(x)}|/M} \leq Cr^n, \]
for some positive \( p_0 \) and \( C \) depending only on \( n \).

For the proof see [4].

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Theorem 4.7 (Weak Harnack Inequality). Suppose that $a_{ij} \in L^\infty(\Omega)$ with $\lambda|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2$ for some $0 < \lambda \leq \Lambda < \infty$ and suppose that $u \in W^{1,2}(\Omega)$ is a non-negative supersolution of the equation $D_j(a_{ij}D_iu) = 0$ in $\Omega$, that is

$$
\int_{\Omega} a_{ij}D_iuD_j\phi \geq 0 \quad \forall \phi \in W^{1,2}_0(\Omega) \quad \phi \geq 0.
$$

(4.16)

Then for all $B_R \subset \Omega$, $0 < p < n/(n-2)$ and $0 < \theta < \tau < 1$ we have

$$
\inf_{B_{R\tau}} u \geq C \left( \frac{1}{R^n} \int_{B_{R\tau}} u^p \right)^{\frac{1}{p}},
$$

(4.17)

where $C = C(n, p, \lambda, \Lambda, \theta, \tau)$.

Proof. We will prove the case $R = 1$. As with Local Boundness (Theorem 4.3) the general result will follow from a scaling argument. The proof proceeds in two steps.

Step 1: We show first that the statement holds for some $p_0 > 0$. Set $\overline{u} := u + k$ for some $k > 0$ so that $\overline{u} > 0$, and define $v := \overline{u}^{-1}$. We start by deriving an equation for $v$. Let $\phi \in W^{1,2}_0(B_1)$ with $\phi \geq 0$ and use $\overline{u}^{-2}\phi$ as the test function in (4.16). We have

$$
\int_{B_1} a_{ij} \overline{u}^{-2}D_iuD_j\phi - 2\int_{B_1} a_{ij}\phi \overline{u}^{-3}D_iuD_j \overline{u} \geq 0.
$$

We note that $D \overline{u} = Du$ and $Dv = -\overline{u}^{-2}D\overline{u}$ and therefore since $a_{ij} \geq 0$

$$
\int_{B_1} a_{ij}D_iuD_j\phi \leq 0.
$$

(4.17)

Applying Theorem 4.5 we get

$$
\sup_{B_{\theta R}} v \leq C\|v\|_{L^p(B_{\tau})}
$$

where $C = C(\theta, \tau, p, n, \lambda, \Lambda) > 0$. In other words

$$
\inf_{B_0} \overline{u} \geq C \left( \int_{B_{\tau}} \overline{u}^{-p} \right)^{-\frac{1}{p}} = C \left( \int_{B_{\tau}} \overline{u}^{-p} \int_{B_{\tau}} \overline{u}^p \right)^{-\frac{1}{p}} \left( \int_{B_{\tau}} \overline{u}^{p} \right)^{\frac{1}{p}}.
$$

If we can show that there exists some $p_0 > 0$ such that

$$
\int_{B_{\tau}} \overline{u}^{-p_0} \int_{B_{\tau}} \overline{u}^{p_0} \leq C(n, \lambda, \Lambda, \theta)
$$

(4.18)

then (4.17) holds in the case $p = p_0$. We will show that there is some $p_0 > 0$ such that for all $\tau < 1$

$$
\int_{B_{\tau}} e^{p_0|w|} \leq C(n, \lambda, \Lambda, \theta),
$$

(4.19)

where $w := \log \overline{u} - |B_{\tau}|^{-1} \int_{B_{\tau}} \log \overline{u}$. From this (4.18) will follow since

$$
\int_{B_{\tau}} \overline{u}^{p_0} \int_{B_{\tau}} \overline{u}^{-p_0} = \int_{B_{\tau}} e^{p_0 \log \overline{u}} \int_{B_{\tau}} e^{-p_0 \log \overline{u}} = \int_{B_{\tau}} e^{p_0(w + |B_{\tau}|^{-1} \int_{B_{\tau}} \log \overline{u})} \int_{B_{\tau}} e^{-p_0(w + |B_{\tau}|^{-1} \int_{B_{\tau}} \log \overline{u})} \leq C e^{B_{\tau}^{-1} \int_{B_{\tau}} \log \overline{u}} \int_{B_{\tau}} e^{-p_0(w + |B_{\tau}|^{-1} \int_{B_{\tau}} \log \overline{u})} \leq C(n, \lambda, \Lambda, \theta).
$$
We first derive an equation for $w$. Choose the test function $u^{-1}\phi$ in (4.16) where $\phi \in L^\infty(B_1) \cap W^{1,2}_0(B_1)$. Then since $Dw = \pi^{-1}D\pi$ we have

$$\int_{B_1} a_{ij} \phi D_i w D_j w \leq \int_{B_1} a_{ij} D_i w D_j \phi \quad \forall \phi \in L^\infty(B_1) \cap W^{1,2}_0(B_1) \quad \phi \geq 0.$$ 

On replacing $\phi$ with $\phi^2$, and using ellipticity, Cauchy-Schwarz inequality and Hölder’s inequality we get

$$\int_{B_1} |Dw|^2 \phi^2 \leq \frac{1}{\lambda} \int_{B_1} a_{ij} \phi^2 D_i w D_j w \leq \frac{2}{\lambda} \int_{B_1} a_{ij} \phi D_i w D_j \phi \leq C \int_{B_1} |Dw| |\phi| |D\phi| \leq C \left( \int_{B_1} |Dw|^2 \zeta^2 \right)^{\frac{1}{2}} \left( \int_{B_1} |D\phi|^2 \right)^{\frac{1}{2}}.$$ 

and hence, in particular,

$$\int_{B_1} |Dw|^2 \zeta^2 \leq C \int_{B_1} |D\phi|^2 \quad \forall \zeta \in C^1_c(B_1).$$

For $B_{2r}(y) \subset B_1$ let $\zeta$ be such that $\text{supp} \zeta \subset B_{2r}(y)$, $\zeta \equiv 1$ in $B_r(y)$ and $|D\zeta| \leq 2/r$, then

$$\int_{B_r(y)} |Dw|^2 \leq Cr^{n-2}.$$ 

Therefore by Hölder’s inequality and Poicaré’s inequality

$$\frac{1}{r^n} \int_{B_r(y)} |w - w_{B_r(y)}| \leq \frac{1}{r^{n/2}} \left( \int_{B_r(y)} |w - w_{B_r(y)}|^2 \right)^{\frac{1}{2}} \leq \frac{1}{r^{n/2}} \left( r^2 \int_{B_r(y)} |Dw|^2 \right)^{\frac{1}{2}} \leq C.$$ 

That is, $w \in \text{BMO}$ and so by the John-Nirenberg lemma

$$\int_{B_r} e^{p_0|w|} \leq C.$$ 

(4.20) 

**Step 2:** We next show that the result holds for all $p < n/(n - 2)$. Note that for $0 < p \leq p_0$ we have $e^{p|w|} \leq e^{p_0|w|}$ and so we already have that the result holds for $p < p_0$. Extending this result requires the use of the Moser iteration technique and the ideas are similar to those in the proof of Theorem 4.3. We give an outline of the argument here. We wish to show that for all $0 < r_1 < r_2 < 1$ and $0 < p_2 < p_1 < n/(n - 2)$ that

$$\left( \int_{B_{r_1}} \pi^{p_1} \right)^{\frac{1}{p_1}} \leq C \left( \int_{B_{r_2}} \pi^{p_2} \right)^{\frac{1}{p_2}}$$

(4.21)

for $C = C(n, \lambda, \Lambda, r_1, r_2, p_1, p_2) > 0$. Choose the test function $\phi = \pi^{-\beta} \eta^2$ where $\beta \in (0, 1)$ and $\eta \in C^1_c(B_1)$. Then plugging this into (4.16) we have

$$0 \leq \int_{B_1} a_{ij} D_i u D_j (\pi^{-\beta} \eta^2) = -\beta \int_{B_1} a_{ij} D_i u \eta^2 + 2 \int_{B_1} a_{ij} D_i u \pi^{-\beta} \eta D_j \eta$$

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implying by ellipticity, Cauchy-Schwarz inequality and Hölder’s inequality that
\[
\beta \int_{B_1} |D \overline{u}|^2 \overline{u}^{-\beta - 1} \eta^2 \leq C \int_{B_1} a_{ij} D_i \overline{u} D_j \overline{u} \overline{u}^{-\beta - 1} \eta^2 \leq C \int_{B_1} a_{ij} D_i \overline{u} \overline{u}^{-\beta} \eta D_j \eta
\]
\[
\leq C \int_{B_1} |D \overline{u}| \overline{u}^{-\beta} |D \eta| \leq \left( \int_{B_1} |Du|^2 \overline{u}^{-\beta - 1} \eta^2 \right)^{\frac{1}{2}} \left( \int_{B_1} |D \eta|^2 \overline{u}^{1-\beta} \right)^{\frac{1}{2}},
\]
and hence
\[
\int_{B_1} |D \overline{u}|^2 \overline{u}^{-\beta - 1} \eta^2 \leq \frac{C \beta^2}{\beta^2} \int_{B_1} |D \eta|^2 \overline{u}^{1-\beta}.
\]
Let \( \gamma := 1 - \beta \in (0, 1) \) and \( w := \overline{u}^{\gamma/2} \), then \( Dw = \frac{\gamma}{2} \overline{u}^{\gamma/2-1} \) and so
\[
\int_{B_1} |Dw|^2 \eta^2 \leq \frac{C}{(1-\gamma)^2} \int_{B_1} |D\eta|^2 w^2
\]
and thus
\[
\int_{B_1} |D(w\eta)|^2 \leq \frac{C}{(1-\gamma)^2} \int_{B_1} w^2 |D\eta|^2.
\]
As in the proof of Theorem 4.3, Sobolev embedding theorem and an appropriate choice of cut-off function imply, with \( \chi = n/(n - 2) \) gives us that for \( 0 < r < R < 1 \)
\[
\left( \int_{B_r} \overline{u}^{\gamma \chi} \right)^{\frac{1}{\gamma \chi}} \leq \left( \frac{C}{(1-\gamma)^2 (R-r)^2} \right)^{\frac{1}{\gamma}} \left( \int_{B_R} \overline{u}^{\chi} \right)^{\frac{1}{\gamma}},
\]
for any \( \gamma \in (0, 1) \). Hence we can obtain (4.17) for \( R = 1 \) after finitely many iterations. \( \square \)

The following result is now an immediate corollary of Theorems 4.5 and 4.7.

**Theorem 4.8** (Moser’s Harnack Inequality). Let \( u \in W^{1,2}(\Omega) \) be a non-negative solution of (4.8) in \( \Omega \), that is
\[
\int_{\Omega} a_{ij} D_i u D_j \phi = 0 \quad \forall \phi \in W^{1,2}_0(\Omega).
\]
Then for any ball \( B_R \subset \Omega \)
\[
\sup_{B_R} u \leq C \inf_{B_{R/2}} u, \quad (4.22)
\]
where \( C = C(n, \lambda, \Lambda) > 0 \).

### 4.2.4 Hölder Continuity of the Derivative

In this section, we will use the Harnack inequalities proved in the last section to show Hölder continuity of solutions. We first state a simple technical lemma.

**Lemma 4.9.** Let \( \tau < 1, \gamma > 0 \) and suppose \( \omega \) is non-decreasing on \( (0, R] \) and satisfies
\[
\omega(\tau r) \leq \gamma \omega(r)
\]
for all \( r \leq R \). Then for all \( r \leq R \)
\[
\omega(r) \leq C \left( \frac{r}{R} \right)^{\alpha} \omega(R)
\]
for constants \( C = C(\gamma, \tau) \) and \( \alpha = \alpha(\gamma, \tau) \). In fact we have \( \alpha = \frac{\log \gamma}{\log \tau} \).
Proof. Let \( r \leq R \) and choose \( k \) such that \( \tau^k R < r \leq \tau^{k-1} R \). Then

\[
\omega(r) \leq \omega(\tau^{k-1} R) \leq \gamma^{k-1} \omega(R).
\]

Moreover we have

\[
\gamma^k = (r^{k})^{\frac{\log \gamma}{\log \tau}} \leq \left( \frac{r}{R} \right)^{\frac{\log \gamma}{\log \tau}}.
\]

Hence

\[
\omega(r) \leq \frac{1}{\gamma} \left( \frac{r}{R} \right)^{\frac{\log \gamma}{\log \tau}} \omega(R).
\]

\[\square\]

**Theorem 4.10.** Let \( u \in W^{1,2}(\Omega) \) be a weak solution in \( \Omega \) of (4.8), that is

\[
\int_{\Omega} a_{ij} D_i u D_j \phi = 0
\]

for all \( \phi \in W^{1,2}_0(\Omega) \). Then \( u \in C^{0,\alpha}(\Omega) \) for \( \alpha \in (0,1) \) depending only on \( n, \lambda \) and \( \Lambda \), and moreover, for any \( B_R \subset \Omega \),

\[
|u(x) - u(y)| \leq C \left( \frac{|x - y|}{R} \right)^{\alpha} \left( \frac{1}{R^n} \int_{B_R} u^2 \right)^{\frac{1}{2}},
\]

for all \( x, y \in B_{\frac{r}{2}} \).

**Proof.** We prove the result for \( R = 1 \). Fix some ball \( B_1 \subset \Omega \) and let \( r \in (0,1] \). Define \( M(r) := \sup_{B_r} u \) and \( m(r) := \inf_{B_r} u \). Then by Theorem 4.5, we know that \( M(r) < \infty \) and \( m(r) > -\infty \). It suffices to show that

\[
\omega(r) := M(r) - m(r) \leq C r^\alpha \left( \int_{B_1} u^2 \right)^{\frac{1}{2}},
\]

for any \( r \leq \frac{1}{2} \). To do so we observe that \( M(r) - u \) is a non-negative solution of (4.8) in \( B_r \), and so Moser’s Harnack inequality (Theorem 4.8) implies

\[
\sup_{B_{\frac{r}{2}}} (M(r) - u) \leq \sup_{B_r} (M(r) - u) \leq C \inf_{B_{\frac{r}{2}}} (M(r) - u).
\]

Thus,

\[
M(r) - m\left( \frac{r}{2} \right) \leq C \left( M(r) - M\left( \frac{r}{2} \right) \right).
\]

Similarly, we can apply Moser’s Harnack inequality to \( u - m(r) \) on \( B_r \) to obtain

\[
M\left( \frac{r}{2} \right) - m(r) \leq C \left( m\left( \frac{r}{2} \right) - m(r) \right).
\]

Adding these two inequalities we obtain

\[
\omega(r) + \omega\left( \frac{r}{2} \right) \leq C \left( \omega(r) - \omega\left( \frac{r}{2} \right) \right),
\]

which implies

\[
\omega\left( \frac{r}{2} \right) \leq \gamma \omega(r),
\]

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where $\gamma = \frac{C-1}{C+1} < 1$. We apply Lemma 4.9 to obtain.

$$\omega(\rho) \leq C\rho^\alpha \omega\left(\frac{1}{2}\right),$$

for all $\rho \in (0, 1/2]$ where $\alpha = \frac{\log \gamma}{\log 2}$. Finally by Theorem 4.3 we have

$$\omega\left(\frac{1}{2}\right) \leq C\left(\int_{B_1} u^2\right)^{\frac{1}{2}},$$

from which we deduce the result. \qed

### 4.3 Hölder Continuity of the Second Derivatives

In this section we will show that the solution $\tilde{u}$ is in $C^{2,\alpha}$, and hence is a solution of the minimal surface equation in the classical sense. The results of this section may be found in [7].

**Lemma 4.11.** Let $\sigma(r)$ be a non-negative monotonically increasing function defined on $[0, R_0]$ satisfying

$$\sigma(r) \leq \gamma \left(\left(\frac{r}{R}\right)^\mu + \delta\right) \sigma(R) + \kappa R^\nu$$

for all $0 < r \leq R \leq R_0$, where $\gamma, \nu < \mu$ and $\delta \geq 0$ are constants. Then there exists $\delta_0(\gamma, \mu, \nu) > 0$ such that, provided $\delta \leq \delta_0$, we have

$$\sigma(r) \leq \gamma_1 \left(\frac{r}{R}\right)^\nu \sigma(R) + \kappa_1 R^\nu$$

for all $0 < r \leq R \leq R_0$ and where $\gamma_1 = \gamma_1(\gamma, \mu, \nu)$ and $\kappa_1 = \kappa_1(\gamma, \mu, \nu, \kappa)$ with $\kappa_1 = 0$ if $\kappa = 0$.

**Proof.** Let $0 < \tau < 1$, and $R < R_0$. Then by assumption

$$\sigma(\tau R) \leq \gamma \tau^\mu (1 + \delta \tau^{-\mu}) \sigma(R) + \kappa R^\nu.$$

We choose $\tau$ such that $2\gamma \tau^\mu = \tau^\lambda$ for some $\nu < \lambda < \mu$, (note that we may assume without loss of generality that $2\gamma > 1$). Assume that $\delta_0 \tau^{-\mu} \leq 1$. It follows that if $\delta \leq \delta_0$

$$\sigma(\tau R) \leq \tau^\lambda \sigma(R) + \kappa R^\nu,$$

and thus, by iterating, that

$$\sigma(\tau^k R) \leq \tau^{k\lambda} \sigma(R) + \kappa \tau^{(k-1)\nu} R^\nu \leq \tau^{k\lambda} \sigma(R) + \kappa \tau^{(k-1)\nu} R^\nu \sum_{j=0}^{k-1} \tau^{j(\lambda-\nu)} \leq \tau^{k\nu} \left(\sigma(R) + \frac{\kappa \tau^{(k-1)\nu} R^\nu}{1 - \tau^{\lambda-\nu}}\right).$$

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Now let $r \leq R$ and choose $k$ such that $\tau^k R < r \leq \tau^{k-1} R$ as in the proof of Lemma 4.9, then

$$
\sigma(r) \leq \sigma(\tau^{k-1} R) \leq \tau^{(k-1)\lambda} \sigma(R) + \frac{\kappa R^\nu \tau^{(k-1)\nu}}{1 - \tau^{(\lambda - \nu)}}
$$

$$
\leq C \tau^{\nu (k-1)} (\sigma(R) + \kappa R^\nu)
$$

$$
\leq \gamma_1 \left( \frac{r}{\tau} \right)^\nu \sigma(R) + \kappa_1 r^\nu,
$$

which completes the proof. \qed

We now state two theorems whose proofs are found in [7], and are needed to show the $C^{2,\alpha}$ regularity. In these theorems $n$ is as usual, the dimension of the space.

**Theorem 4.12** (Campanato’s Theorem). If for all $0 < r \leq R_0$ and all $x \in \Omega$ we have

$$
\int_{B_r(x)} |u - u_{B_r(x)}|^p \leq \gamma r^{n + p\alpha}
$$

with constants $\gamma$, $p$ and $0 < \alpha < 1$, then $u \in C^{0,\alpha}_{\text{loc}}(\Omega)$.

**Theorem 4.13** (Campanato Estimates). Let $(A_{ij})$ be a matrix with $|A_{ij}| \leq \Lambda$ for all $i, j$ and $A_{ij} \xi_i \xi_j \geq \lambda |\xi|^2$ for all $\xi \in \mathbb{R}^n$ where $0 < \lambda \leq \Lambda < \infty$ are constants. Suppose $u \in W^{1,2}(\Omega)$ is a weak solution of the equation

$$
D_j (A_{ij} D_i u) = 0. \quad (4.23)
$$

Then for all $x_0 \in \Omega$ and $0 < r < R < \text{dist}(x_0, \partial \Omega)$ we have

$$
\int_{B_r(x_0)} |u|^2 \leq c_1 \left( \frac{r}{\tau} \right)^n \int_{B_{\tau r}(x_0)} |u|^2
$$

$$
\int_{B_r(x_0)} |u - u_{B_r(x_0)}|^2 \leq c_2 \left( \frac{r}{\tau} \right)^{n+2} \int_{B_{\tau r}(x_0)} |u - u_{B_{\tau r}(x_0)}|^2.
$$

**Theorem 4.14.** Suppose that $a_{ij}(x)$ are $C^\alpha(\Omega)$ for $i, j = 1, \ldots, n$ and some $\alpha \in (0, 1)$ and satisfy $a_{ij}(x) \xi_i \xi_j \geq \lambda |\xi|^2$ for all $\xi \in \mathbb{R}^n$ and $x \in \Omega$, and that $|a_{ij}| \leq \Lambda$ on $\Omega$, where $0 < \lambda \leq \Lambda < \infty$ are constant. Then any weak solution $u \in W^{1,2}(\Omega)$ of the equation

$$
D_j (a_{ij} D_i u) = 0
$$

is $C^{1,\alpha'}(\Omega)$ for all $\alpha' \in (0, \alpha)$.

**Proof.** Step 1: Let $x_0 \in \Omega$. We write

$$
a_{ij}(x) = a_{ij}(x_0) + (a_{ij}(x) - a_{ij}(x_0)) \quad (4.26)
$$

and define $A_{ij} := a_{ij}(x_0)$. Then for $v$ satisfying (4.25)

$$
D_j (A_{ij} D_i u) = D_j ((a_{ij}(x_0) - a_{ij}(x)) D_i u) = D_j f^j(x) \quad (4.27)
$$

where we have introduced the function

$$
f^j(x) := (a_{ij}(x_0) - a_{ij}(x)) D_i u(x).$$
In other words, \( u \) satisfies

\[
\int_{\Omega} A_{ij} D_i v D_j \phi = \int_{\Omega} f^j D_j \phi \quad \forall \phi \in W^{1,2}_0(\Omega). 
\]

(4.28)

Let \( B_R(x_0) \subset \Omega \) be a ball, and let \( w \in W^{1,2}(B_R(x_0)) \) be a weak solution of the Dirichlet problem

\[
D_j(A_{ij} D_i w) = 0 \quad \text{in } B_R(x_0)
\]

\[
w = u \quad \text{on } \partial B_R(x_0).
\]

(4.29)

Then \( w \) satisfies

\[
\int_{B_R(x_0)} A_{ij} D_i w D_j \phi = 0 \quad \forall \phi \in W^{1,2}_0(B_R(x_0)).
\]

(4.30)

Such a \( w \) can be shown to exist by the Lax Milgram Theorem, since it guarantees the existence of \( z \in W^{1,2}_0(B_R(x_0)) \) with

\[
B(\phi, z) = \int_{B_R(x_0)} A_{ij} D_i z D_j \phi = -\int_{B_R(x_0)} A_{ij} D_i u D_j \phi \quad \forall \phi \in W^{1,2}_0(B_R(x_0)),
\]

(4.31)

and then \( w \) given by \( w = u + z \) is easily seen to be a solution of (4.29). According to Theorem 4.2, we may deduce that \( w \) is in \( W^{2,2}_{\text{loc}}(B_R(x_0)) \), and so since the equation for \( w \) is linear with constant coefficients, it follows that \( D_k w \) satisfies the same equation on \( B_R(x_0) \) for \( k = 1, \ldots, n \) although naturally the boundary conditions will be different. Hence, by Campanato estimates (4.24) we get

\[
\int_{B_r(x_0)} |Dw|^2 \leq C \left( \frac{r}{R} \right)^n \int_{B_R(x_0)} |Dw|^2.
\]

(4.32)

**Step 2:** Next we observe that since \( u = w \) on \( \partial B_R(x_0) \), \( u - w \) is an admissible test function, and hence

\[
\int_{B_R(x_0)} A_{ij} D_i w D_j w = \int_{B_R(x_0)} A_{ij} D_i w D_j u
\]

Combining this, the Cauchy-Schwarz inequality and ellipticity we obtain the estimate

\[
\lambda \int_{B_R(x_0)} |Dw|^2 \leq \int_{B_R(x_0)} A_{ij} D_i w D_j w = \int_{B_R(x_0)} A_{ij} D_i w D_j u
\]

\[
\leq \int_{B_R(x_0)} |ADw||Du| \leq \left( \int_{B_R(x_0)} \|A\|^2 |Dw|^2 \right)^{\frac{1}{2}} \left( \int_{B_R(x_0)} |Du|^2 \right)^{\frac{1}{2}}
\]

\[
\leq n\lambda \left( \int_{B_R(x_0)} |Dw|^2 \right)^{\frac{1}{2}} \left( \int_{B_R(x_0)} |Du|^2 \right)^{\frac{1}{2}},
\]

which implies

\[
\int_{B_R(x_0)} |Dw|^2 \leq \left( \frac{n\lambda}{\lambda} \right)^2 \int_{B_R(x_0)} |Du|^2.
\]

(4.33)

Moreover, (4.28) and (4.30) together imply

\[
\int_{B_R(x_0)} A_{ij} D_i (u - w) D_j \phi = \int_{B_R(x_0)} f^j D_j \phi \quad \forall \phi \in W^{1,2}_0(B_R(x_0)).
\]
Letting $\phi = u - w$ we obtain

$$\int_{B_R(x_0)} |D(u - w)|^2 \leq \frac{1}{\lambda} \int_{B_R(x_0)} A_{ij} D_i(u - w) D_j(u - w) = \frac{1}{\lambda} \int_{B_R(x_0)} f_j D_j(u - w)$$

$$\leq \frac{1}{\lambda} \left( \int_{B_R(x_0)} |D(u - w)|^2 \right)^{\frac{1}{2}} \left( \int_{B_R(x_0)} \sum_j |f_j|^2 \right)^{\frac{1}{2}},$$

implying

$$\int_{B_R(x_0)} |D(u - w)|^2 \leq \frac{1}{\lambda^2} \int_{B_R(x_0)} \sum_j |f_j|^2. \tag{4.34}$$

**Step 3:** We now start to combine all of these estimates. First we use (4.32) and (4.33) to show

$$\int_{B_r(x_0)} |Du|^2 \leq 2 \int_{B_r(x_0)} |Dw|^2 + 2 \int_{B_r(x_0)} |D(u - w)|^2$$

$$\leq C \left( \frac{r}{R} \right)^n \int_{B_R(x_0)} |Dw|^2 + 2 \int_{B_r(x_0)} |D(u - w)|^2$$

$$\leq C \left( \frac{r}{R} \right)^n \int_{B_R(x_0)} |Du|^2 + 2 \int_{B_r(x_0)} |D(u - w)|^2.$$

Next observe that

$$\int_{B_r(x_0)} |D(u - w)|^2 \leq \int_{B_r(x_0)} |D(u - w)|^2 \leq \frac{1}{\lambda^2} \int_{B_R(x_0)} \sum_j |f_j|^2$$

$$\leq \frac{1}{\lambda^2} \sup_{i,j \in \Omega} |a_{ij}(x_0) - a_{ij}(x)| \int_{B_R(x_0)} |Du|^2 \leq CR^{2\alpha} \int_{B_r(x_0)} |Du|^2, \tag{4.35}$$

the last inequality following since each $a_{ij}$ is in $C^{0,\alpha}$ by assumption. Finally we arrive at the estimate

$$\int_{B_r(x_0)} |Du|^2 \leq \gamma \left( \left( \frac{r}{R} \right)^n + R^{2\alpha} \right) \int_{B_R(x_0)} |Du|^2.$$

We apply Lemma 4.11 with $\sigma(r) := \int_{B_r(x_0)} |Du|^2$, $\mu = n$, and $\nu = n - \varepsilon$ for some $\varepsilon > 0$. Then there exists some $\delta_0(\gamma, n, \varepsilon) > 0$ such that provided $R_0^{2\alpha} \leq \delta_0$, then for all $0 < r \leq R \leq R_0$

$$\int_{B_r(x_0)} |Du|^2 \leq C \left( \frac{r}{R} \right)^{n-\varepsilon} \int_{B_R(x_0)} |Du|^2, \tag{4.36}$$

where the constant $C$ depends on $\varepsilon$, $n$ and $\gamma$.

**Step 4:** The second part of the proof is devoted to deriving an analogous inequality from the second Campanato estimate (4.24). Using the same arguments as were used to establish (4.32), we can show that

$$\int_{B_r(x_0)} |Dw - (Dw)_{B_r(x_0)}|^2 \leq C \left( \frac{r}{R} \right)^{n+2} \int_{B_R(x_0)} |Dw - (Dw)_{B_R(x_0)}|^2. \tag{4.37}$$
Next we note that
\[
\int_{B_{\rho}(x_0)} |Dw - (Du)_{B_{\rho}(x_0)}|^2 \leq \int_{B_{\rho}(x_0)} |Dw - (Du)_{B_{\rho}(x_0)}|^2,
\]
since for any real-valued \( g \in L^2(B_{\rho}(x_0)) \)
\[
\int_{B_{\rho}(x_0)} |g - (g)_{B_{\rho}(x_0)}|^2 = \inf_{\kappa \in \mathbb{R}} \int_{B_{\rho}(x_0)} (g - \kappa)^2.
\]
To see this note that the function \( F(\kappa) := \int_{B_{\rho}(x_0)} (g - \kappa)^2 \) is convex and differentiable and has a critical point at \( \kappa = (g)_{B_{\rho}(x_0)} \), and so \( F \) attains its minimum here. Furthermore, we also have the following estimate by ellipticity
\[
\int_{B_{\rho}(x_0)} |Dw - (Du)_{B_{\rho}(x_0)}|^2 \leq \frac{1}{\lambda} \int_{B_{\rho}(x_0)} A_{ij}(D_i w - (D_i u)_{B_{\rho}(x_0)})(D_j w - (D_j u)_{B_{\rho}(x_0)})
\]
\[
= \frac{1}{\lambda} \int_{B_{\rho}(x_0)} A_{ij}(D_i w - (D_i u)_{B_{\rho}(x_0)})(D_j u - (D_j u)_{B_{\rho}(x_0)})
\]
\[
+ \frac{1}{\lambda} \int_{B_{\rho}(x_0)} A_{ij}(D_i u)_{B_{\rho}(x_0)}(D_j u - D_j w).
\]
The second integral vanishes since \( A_{ij}(D_i u)_{B_{\rho}(x_0)} \) is constant and \( u - w \in W^{1,2}_0(B_{\rho}) \). Applying the Cauchy-Schwarz inequality as we did to deduce (4.33), we have
\[
\int_{B_{\rho}(x_0)} |Dw - (Du)_{B_{\rho}(x_0)}|^2 \leq \frac{\lambda^2 u_{\nu}^2}{\lambda^2} \int_{B_{\rho}(x_0)} |Du - (Du)_{B_{\rho}(x_0)}|^2. \tag{4.38}
\]

**Step 5:** Finally
\[
\int_{B_{\rho}(x_0)} |Du - (Du)_{B_{\rho}(x_0)}|^2 \leq 3 \int_{B_{\rho}(x_0)} |Du - Dw|^2 + 3 \int_{B_{\rho}(x_0)} |Dw - (Dw)_{B_{\rho}(x_0)}|^2
\]
\[
+ 3 \int_{B_{\rho}(x_0)} |(Du)_{B_{\rho}(x_0)} - (Dw)_{B_{\rho}(x_0)}|^2,
\]
and
\[
\int_{B_{\rho}(x_0)} |(Du)_{B_{\rho}(x_0)} - (Dw)_{B_{\rho}(x_0)}|^2 = |B_{\rho}(x_0)| \left( \frac{1}{|B_{\rho}(x_0)|} \int_{B_{\rho}(x_0)} |Du - Dw| \right)^2
\]
\[
\leq |B_{\rho}(x_0)| \left( \frac{1}{|B_{\rho}(x_0)|} |B_{\rho}(x_0)|^{1/2} \left( \int_{B_{\rho}(x_0)} |Du - Dw|^2 \right)^{1/2} \right)^2
\]
\[
= \int_{B_{\rho}(x_0)} |Du - Dw|^2.
\]
Combining the last two inequalities and using (4.35) gives
\[
\int_{B_{\rho}(x_0)} |Du - (Du)_{B_{\rho}(x_0)}|^2 \leq 3 \int_{B_{\rho}(x_0)} |Dw - (Dw)_{B_{\rho}(x_0)}|^2 + 6 \int_{B_{\rho}(x_0)} |Du - Dw|^2
\]
\[
\leq 3 \int_{B_{\rho}(x_0)} |Dw - (Dw)_{B_{\rho}(x_0)}|^2 + C R^{2\alpha} \int_{B_{\rho}(x_0)} |Du|^2;
\]
33
Applying the second Campanato estimate \([4.24]\) to the first integral and then applying \([4.38]\), we have
\[
\int_{B_r(x_0)} |Du - (Du)_{B_r(x_0)}|^2 \leq C \left( \frac{r}{R} \right)^{n+2} \int_{B_R(x_0)} |Du - (Du)_{B_R(x_0)}|^2 + CR^{2\alpha} \int_{B_R(x_0)} |Du|^2 \\
\leq C \left( \frac{r}{R} \right)^{n+2} \int_{B_R(x_0)} |Du - (Du)_{B_R(x_0)}|^2 + CR^{n+2\alpha-\epsilon}
\]
where we have applied \([4.36]\) with \(r = R\) and \(R = R_0\) to deduce the second inequality. Finally applying Lemma \([4.11]\) we get
\[
\int_{B_r(x_0)} |Dv - (Dv)_{B_r(x_0)}|^2 \leq \left( C_1 \left( \frac{1}{R} \right)^{n+2\alpha-\epsilon} \int_{B_R(x_0)} |Du - (Du)_{B_R(x_0)}|^2 + C_2 \right) r^{n+2\alpha-\epsilon}.
\]
This holds for all \(0 < r \leq R\) and so the claim now follows from Campanato’s Theorem.

We apply the above theorem to the partial derivatives of \(\tilde{u}\) and the equation \([4.7]\) to deduce that \(\tilde{u}\) is \(C^{2,\alpha}\).

### 4.4 Schauder Theory

Having shown that the solution \(\tilde{u}\) is a classical solution, we now appeal to the interior Schauder estimates to show that \(\tilde{u}\) is smooth. Roughly speaking, Schauder estimates may be used to show that the second derivatives of a classical solution, have the same regularity as the coefficients in the equation, so in our case in particular we can show that \(a_{ij} \in C^{k,\alpha}(\Omega)\) implies \(u \in C^{k+2,\alpha}(\Omega)\). Since the coefficients \(a_{ij}\) are written in terms of the first derivatives of \(\tilde{u}\), a bootstrapping argument will allow us to deduce the result we are after. The first two results we shall state without proof, they may be found in, for example, \([2]\).

**Theorem 4.15** (Dirichlet Problem). Let \(B \subset \mathbb{R}^n\) be a ball, and let
\[
L = a_{ij}D_{ij} + b_iD_i + c
\]
be a uniformly elliptic operator with coefficients \(a_{ij}, b_i\) and \(c \in C^{0,\alpha}(\overline{B})\), and \(c \leq 0\) on \(B\). Then the Dirichlet problem
\[
Lu = f \quad \text{in } B \\
u = g \quad \text{on } \partial B,
\]
for \(f \in C^{0,\alpha}(\overline{B})\) and \(g \in C(\partial B)\) has a unique solution \(u \in C^{2,\alpha}(B) \cap C(\overline{\Omega})\).

**Theorem 4.16** (Interior Schauder Estimates). Let \(\alpha \in (0,1)\). Suppose \(\Omega\) is a domain and \(u \in C^{2,\alpha}(\overline{\Omega})\) solves the uniformly elliptic equation
\[
Lu = a_{ij}D_{ij}u + b_iD_iu + cu = f \quad \text{in } \Omega,
\]
where \(a_{ij}, b_i, c\) and \(f\) are in \(C^{0,\alpha}(\overline{\Omega})\). Then for any \(\Omega' \subset \subset \Omega\)
\[
\|u\|_{C^2,\alpha(\Omega')} \leq C(\|u\|_{C(\Omega)} + \|f\|_{C^{0,\alpha}(\Omega)})
\]
where \(C = C(n,\alpha,L,\Omega',\Omega) \in (0,\infty)\).
Theorem 4.15 (Interior Regularity). Let $k \geq 0$ be an integer and $\alpha \in (0, 1)$. Let $\Omega \subset \mathbb{R}^n$ be open and suppose that $u \in C^2(\Omega)$ satisfies

$$Lu = a_{ij}D_{ij} + b_iD_iu + cu = f \quad \text{in } \Omega$$

for some elliptic operator $L$ with coefficients $a_{ij}$, $b_i$, $c \in C^{k,\alpha}(\Omega)$ and some $f \in C^{k,\alpha}(\Omega)$. Then $u \in C^{k+2,\alpha}(\Omega)$.

**Proof.** We will prove the simpler case where the operator $L$ takes the form $L = a_{ij}D_{ij}$. The more general result is proved in a similar fashion with slightly more complicated calculations.

**Step 1:** We first show that $a_{ij}$, $f \in C^{0,\alpha}(\Omega)$ and $u \in C^2(\Omega)$ implies that $u \in C^{2,\alpha}(\Omega)$. This is a direct consequence of Theorem 4.15. Let $B \subset \subset \Omega$ be a ball and consider solutions $v$ to the Dirichlet problem

$$a_{ij}D_{ij}v = f \quad \text{in } B,$$

$$v = u \quad \text{on } \partial B.$$

Theorem 4.15 tells us that there exists a solution in $C(\overline{B}) \cap C^{2,\alpha}(B)$, however we also know that $u$ is a $C(\overline{B}) \cap C^2(B)$ solution and that there exists at most one solution in $C(\overline{B}) \cap C^2(B)$. Thus we conclude that $u \in C^{2,\alpha}(B)$ and hence that $u \in C^{2,\alpha}$.

**Step 2:** We next show that $a_{ij}$, $f \in C^{1,\alpha}(\Omega)$ and $u \in C^{2,\alpha}(\Omega)$ implies that $u \in C^{3,\alpha}(\Omega)$. This is done via a difference quotient argument. Choose $\Omega'' \subset \subset \Omega'$ such that $\partial \Omega'' \subset \subset \Omega' \subset \subset \Omega$ and $h_0 > 0$ such that both $\text{dist}(\Omega'', \partial \Omega'') < h_0$ and $\text{dist}(\Omega'', \partial \Omega') < h_0$. We then apply $\delta_h^i$ for some $l \in \{1, \ldots, n\}$ and $0 < h < h_0$ to both sides of the equation $Lu = f$ and rearrange to get

$$L(\delta_h^i u) = a_{ij}(x)D_{ij}(\delta_h^i u) = \delta_h^i f - \delta_h^i a_{ij}(x)D_{ij}u(x + he_l) \quad \text{on } \Omega''.$$
theorem we can find a sequence \((h_j)_{j \in \mathbb{N}}\) with \(h_j \searrow 0\) such that \(\delta_i^{h_j} u\) converges in \(C^2(\overline{\Omega})\). Now
\[ u \text{ is continuously differentiable on } \Omega \text{ and so } \delta_i^{h_j} u \text{ converges uniformly to } D_i u \text{ on } \Omega'' \]. Thus we conclude that \(\delta_i^{h_j} u\) converges to \(D_i u\) in \(C^2(\overline{\Omega})\), which implies \(D_i u \in C^2,\alpha(\overline{\Omega})\). Since \(l\) and \(\Omega''\) were arbitrary, we conclude that \(u \in C^{3,\alpha}(\Omega)\).

**Step 3:** Let \(k \geq 2\). We now wish to show that \(a_{ij}, f \in C^{k,\alpha}(\Omega)\) and \(u \in C^{k+1,\alpha}(\Omega)\) implies \(u \in C^{k+2,\alpha}(\Omega)\). Let \(\beta\) be a multiindex with \(|\beta| = k - 1\). Then applying \(D^\beta\) to both sides of the equation \(Lu = f\), we obtain
\[ L(D^\beta u) = D^\beta f - \sum_{\gamma < \beta} \frac{\beta!}{\gamma!(\beta - \gamma)!} D^{\beta - \gamma} a_{ij} D^\gamma D_{ij} u \quad \text{in } \Omega. \]
Since the right hand side is in \(C^{1,\alpha}(\Omega)\) and \(D^\beta u \in C^{2,\alpha}\), we may apply step 2 to deduce that \(D^\beta u \in C^{3,\alpha}(\Omega)\), and hence that \(u \in C^{k+2,\alpha}\). The proof of the theorem now follows by induction.

5 **Gradient bound**

In this section, we want to find a bound of the gradient of a solution \(u\) to the minimal surface equation. We will state the main theorem here, but we will have to do a bit of work before we can prove it. This section is based on [6], and any proofs omitted are proved there or in [2].

**Theorem 5.1** (Gradient Bound). Let \(\Omega\) be a bounded \(C^2\) domain and suppose that \(u\) solves the minimal surface equation in \(\Omega\). Then for any \(y \in \Omega\), we have the following bound on gradient of \(u\):
\[ |Du(y)| \leq \exp \left( C \left( 1 + \sup_{x \in B_d(y)} \left[ \frac{u(x) - u(y)}{d} \right] \right) \right), \]
where \(C = C(u)\) and \(d = \text{dist}(y, \partial \Omega)\).

We will split the proof up into a series of propositions, but first of all we will define our notation and present a few useful results. We assume throughout this section, as we did in the statement of Theorem 5.1, that \(\Omega \subset \mathbb{R}^n\) is a bounded \(C^2\) domain and that \(u\) solves the minimal surface equation in \(\Omega\). We will write \(G(u)\) for the graph of \(u\), namely
\[ G(u) = \{(x, u(x)) : x \in \Omega\} \]
and write \(a = \sqrt{1 + |Du|^2}\) for the area element of \(u\) on its graph. The graph can equivalently be viewed as a level set of the function \(\Phi : \Omega \times \mathbb{R} \to \mathbb{R}\) defined by \(\Phi(x, x_{n+1}) = x_{n+1} - u(x)\). The unit normal to \(G(u)\) is given by
\[ \nu = \frac{D\Phi}{|D\Phi|}. \quad (5.1) \]

**Lemma 5.2.** If \(\nu\) is the normal to \(G(u)\) as defined in [5.1], then
\[ \text{div}(\nu) := D_i \nu_i = -nH, \]
where \(H\) is the mean curvature of the graph \(G(u)\).
Lemma 5.3. Let $u$ be a solution to the minimal surface equation, $\phi \in C^1_c(\Omega)$ a compactly supported, differentiable test function, and $\nu$ the normal to $\mathcal{G}(u)$ as defined in equation (5.1). Then

$$\int_{\Omega} \nu_i D_i \phi = 0.$$  

Proof. Since $u$ is a solution to the minimal surface equation, we know that it has mean curvature, $H$, equal to zero. Therefore, by Lemma 5.2, we know that

$$\text{div } \nu = D_i \nu_i = 0 \text{ in } \Omega.$$  

We then multiply by $\phi \in C^1_c(\Omega)$, integrate over $\Omega$, and integrate by parts to finish the proof. 

Definition (Tangential gradient). For a differentiable function $g$ defined in some open set containing the graph $\mathcal{G}(u)$, we define the tangential gradient of $g$ on $\mathcal{G}(u)$ to be

$$\nabla g = Dg - (Dg \cdot \nu)\nu.$$  

Note that for $y \in \mathcal{G}(u)$, the tangential gradient $\nabla g(y)$ is the projection of $Dg(y)$ onto the tangent plane to $\mathcal{G}(u)$ at $y$.

Lemma 5.4. Let $u$ be a solution to the MSE, and suppose that we have a function $\tilde{\phi} \in C^1_c(\Omega \times \mathbb{R})$. Define the function $\phi : \Omega \to \mathbb{R}$ by setting

$$\phi(x') = \tilde{\phi}(x', u(x')),$$

where $x' = (x_1, \ldots, x_n) \in \Omega$. Then on $\mathcal{G}(u)$ we have

$$\nabla_i \nu_{n+1} D_i \phi = \nabla_i \nu_{n+1} \nabla_i \phi = \nabla_i \nu_{n+1} \nabla_i \tilde{\phi}.$$  

Proposition 5.5. Let $w = \log \sqrt{1 + |Du|^2}$. Then for any $y \in \Omega$, and $\rho > 0$ such that $B^n_\rho(y) \subset \Omega$, we have the following inequality:

$$w(y) \leq \frac{1}{\omega_n \rho^n} \int_{\mathcal{G}(u) \cap B^{n+1}_\rho(y)} w \, d\mathcal{H}^n.$$  

We split the proof of this proposition up into two lemmas.

Lemma 5.6. If $u$ is a solution to the MSE, then the function $w = \log \sqrt{1 + |Du|^2}$ is subharmonic, in the sense that

$$\nabla_i \nabla_i w \geq 0.$$
Proof. From Lemma 5.3, we know that
\[ \int_{\Omega} \nu_i D_i \phi = 0, \]
where \( \nu \) is the tangent to \( G(u) \), for any \( \phi \in C^1_c(\Omega) \). Therefore, for any function \( \phi \) such that \( D_k \phi \in C^1_c(\Omega) \) for some \( k = 1, \ldots, n+1 \), we can integrate by parts to see that
\[ \int_{\Omega} D_k \nu_i D_i \phi = 0. \]
Now let us choose \( \phi \) so that it is non-negative in \( \Omega \), and \( D_i(\nu_k \phi) \in C^1_c(\Omega) \) for each \( k, i = 1, \ldots, n+1 \). Then we can use \( \nu_k \phi \) as our test functions in the equation above and sum over \( k \) to get
\[ \int_{\Omega} D_k \nu_i D_i (\nu_k \phi) = 0, \]
and therefore
\[ \int_{\Omega} (D_k \nu_i) (D_i \nu_k) \phi + \int_{\Omega} \nu_k (D_k \nu_i) (D_i \phi) = 0. \tag{5.2} \]
It can be shown that
\[ D_k \nu_i D_i \nu_k = \sum_k (\kappa_k)^2 \geq 0, \]
where \( \kappa_k \) are the principal curvatures, and that
\[ \nu_k D_k \nu_i D_i = -a \nabla_i \nu_{n+1}. \]
Substituting these into (5.2), we can see that
\[ \int_{\Omega} \left( \sum_k (\kappa_k)^2 \right) \phi - \int_{\Omega} a \nabla_i \nu_{n+1} D_i \phi = 0. \]
Therefore, since we have chosen \( \phi \) so that it is non-negative in \( \Omega \), we now know that
\[ \int_{\Omega} a \nabla_i \nu_{n+1} D_i \phi \geq 0. \tag{5.3} \]
Now, if we find a function \( \tilde{\phi} \in C^1_c(\Omega \times \mathbb{R}) \) such that \( \phi(x) = \tilde{\phi}(x, u(x)) \), then by Lemma 5.4 we have
\[ \int_{\Omega} a \nabla_i \nu_{n+1} D_i \phi = \int_{G(u)} \nabla_i \nu_{n+1} \nabla_i \tilde{\phi} \, d\mathcal{H}^n. \]
Combining this with (5.3) gives
\[ \int_{G(u)} \nabla_i \nu_{n+1} \nabla_i \tilde{\phi} \, d\mathcal{H}^n \geq 0. \tag{5.4} \]
To move on from here note that, by the definition of the normal vector \( \nu \) and the area element \( a \), we have that \( \nu_{n+1} = 1/a \). If we also use a test function \( \tilde{\phi} \) in (5.4) which has the from \( \tilde{\phi} = a\varphi \), we can re-write the equation as:

\[
0 \leq \int_{\mathcal{G}(u)} \nabla_i \nu_{n+1} \nabla_i \tilde{\phi} d\mathcal{H}^n \\
= - \int_{\mathcal{G}(u)} \frac{1}{a^2} \nabla_i a \nabla_i (a \varphi) d\mathcal{H}^n \\
= - \int_{\mathcal{G}(u)} \frac{1}{a^2} \nabla_i a \nabla_i a \varphi d\mathcal{H}^n - \int_{\mathcal{G}(u)} \frac{1}{a} \nabla_i a \nabla_i \varphi d\mathcal{H}^n \\
= - \left[ \int_{\mathcal{G}(u)} \frac{|\nabla a|^2 \varphi}{a^2} + \frac{\nabla a \cdot \nabla \varphi}{a} d\mathcal{H}^n \right].
\]

We have seen before that \( w = \log a \), which means that \( \nabla w = \frac{1}{a} \nabla a \). Therefore, the above inequality can be re-written as

\[
\int_{\mathcal{G}(u)} |\nabla w|^2 \varphi + \nabla w \cdot \nabla \varphi d\mathcal{H}^n \leq 0. \tag{5.5}
\]

We now use the integration by parts formula on the graph \( \mathcal{G}(u) \),

\[
\int_{\mathcal{G}(u)} \nabla f \cdot \nabla g d\mathcal{H}^n = - \int_{\mathcal{G}(u)} g \Delta f d\mathcal{H}^n,
\]

to see that

\[
\int_{\mathcal{G}(u)} (|\nabla w|^2 - \Delta w) \varphi d\mathcal{H}^n \leq 0.
\]

Therefore, since we have implicitly assumed that \( \varphi \) is positive on \( \mathcal{G}(u) \), we can conclude that

\[
\Delta w \geq |\nabla w|^2 \geq 0,
\]

i.e. the function \( w = \log \sqrt{1 + |Du|} \) is subharmonic on the graph of \( u \).

\[\square\]

**Lemma 5.7** (Mean Value Inequality). Suppose that \( w \in C^2(\Omega \times \mathbb{R}) \) is a non-negative, subharmonic function on \( \mathcal{G}(u) \). Then for any \( y \in \mathcal{G}(u) \) and any \( \rho > 0 \) such that \( B^\rho(y) \subset \Omega \), we have that

\[
w(y) \leq \frac{1}{\omega_n \rho^n} \int_{\mathcal{G}(u) \cap B^\rho(y)} w d\mathcal{H}^n,
\]

where \( \omega_n \) is the measure of the \( n \)-dimensional unit ball.

**Proof of Proposition 5.5**. All of the hard work has now been done: we use Lemma 5.6 to see that the function \( w = \log \sqrt{1 + |Du|^2} \) is subharmonic, which means that we can apply Lemma 5.7 to conclude that we have the inequality

\[
w(y) \leq \frac{1}{\omega_n \rho^n} \int_{\mathcal{G}(u) \cap B^\rho(y)} w d\mathcal{H}^n,
\]

for any \( y \in \mathcal{G}(u) \) and \( \rho > 0 \) such that \( B^\rho(y) \subset \Omega \).

\[\square\]
Our aim now is to bound the integral on the right hand side of the inequality in Proposition 5.5.

**Proposition 5.8.** For any \( y \in \Omega \), and \( \rho > 0 \) such that \( B_{\rho}^{n+1}(y) \subset \Omega \times \mathbb{R} \), we have the following inequality:

\[
\int_{\mathcal{G}(u) \cap B_{\rho}^{n+1}(y)} w \, d\mathcal{H}^n \leq \omega_n \rho^n + \int_{B_{\rho}^{n}(y) \cap \{|u-u(y)|<\rho\}} \frac{|Du|^2}{a} w,
\]

where \( \omega_n \) is the measure of the \( n \) dimensional unit ball.

**Proof.** Since \( w \geq 0 \) in \( \mathcal{G}(u) \), and \( B_{\rho}^{n+1}(y) \subset B_{\rho}^{n}(y) \times \{|x_{n+1} - u(y)| < \rho\} \), we know that

\[
\int_{\mathcal{G}(u) \cap B_{\rho}^{n+1}(y)} w \, d\mathcal{H}^n \leq \int_{\mathcal{G}(u) \cap (B_{\rho}^{n}(y) \times \{|x_{n+1} - u(y)| < \rho\})} w \, d\mathcal{H}^n.
\]

By definition of \( \mathcal{H}^n \), we can re-write the right hand integral above as

\[
\int_{\mathcal{G}(u) \cap (B_{\rho}^{n}(y) \times \{|x_{n+1} - u(y)| < \rho\})} w \, d\mathcal{H}^n = \int_{B_{\rho}^{n}(y) \cap \{|u-u(y)|<\rho\}} aw.
\]

Now, recall that \( a = \sqrt{1 + |Du|^2} \), which means we can write \( a = (1 + |Du|^2)/a \), and also \( w = \log a \). Therefore,

\[
\int_{B_{\rho}^{n}(y) \cap \{|u-u(y)|<\rho\}} aw = \int_{B_{\rho}^{n}(y) \cap \{|u-u(y)|<\rho\}} \frac{1 + |Du|^2}{a} w \leq \int_{B_{\rho}^{n}(y) \cap \{|u-u(y)|<\rho\}} \frac{|Du|^2}{a} w, \quad \text{since} \quad \frac{\log a}{a} \leq 1
\]

\[
\leq \omega_n \rho^n + \int_{B_{\rho}^{n}(y) \cap \{|u-u(y)|<\rho\}} \frac{|Du|^2}{a} w,
\]

which completes the proof. \( \square \)

Again, we want to bound the integral on the right hand side of the inequality in Proposition 5.8.

**Proposition 5.9.** For any \( y \in \Omega \), and \( \rho > 0 \) such that \( B_{2\rho}^{2n}(y) \subset \Omega \), we have the following inequality:

\[
\int_{B_{\rho}^{n}(y) \cap \{|u-u(y)|<\rho\}} \frac{|Du|^2}{a} w \leq C \int_{B_{2\rho}^{n}(y) \cap \{|u-u(y)|>2\rho\}} a,
\]

where \( C = C(u) \) is independent of the choice of \( y \) and \( \rho \).

As before, we will split the proof up into some lemmas.

**Lemma 5.10.** Let \( y \in \Omega \), and find \( \rho > 0 \) such that \( B_{2\rho}^{n}(y) \subset \Omega \). Let \( \eta \in C_0^1(\Omega) \) be a function satisfying

1) \( 0 \leq \eta \leq 1 \) in \( \Omega \),
2) \( |D\eta| \leq 2/\rho \) in \( \Omega \),

\( 40 \)
3) $\eta = 1$ in $B^n_\rho(y)$,

4) $\eta = 0$ in $\Omega / B^n_\rho(y)$.

Then the following inequality holds:

$$
\int_{B^n_\rho(y) \cap \{|u-u(y)|<\rho\}} \frac{|Du|^2}{a} w \leq 4 \int_{B^n_{2\rho}(y) \cap \{|u-u(y)|>\rho\}} a + 2\rho \int_{B^n_{2\rho}(y) \cap \{|u-u(y)|>\rho\}} \eta|Dw|.
$$

**Proof.** Without loss of generality, we assume that $y = 0$ and $u(y) = 0$. Now, recall the result in Lemma 5.3, that

$$
\int_\Omega \nu_i D_i \phi = 0,
$$

(5.6)

for $\phi \in C^1_c(\Omega)$. We will prove the inequality in this lemma by choosing the test function $\phi$ carefully.

Let $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by

$$
\gamma(t) = \begin{cases} 
0 & \text{for } t < -\rho, \\
t + \rho & \text{for } -\rho \leq t \leq \rho, \\
2\rho & \text{for } t > \rho.
\end{cases}
$$

Then, we let our test function $\phi$ in (5.6) be $\phi = \eta w \gamma(u)$. Since $D_i(\eta w \gamma(u)) = w \gamma(u) D_i \eta + \eta \gamma(u) D_i w + \eta w \gamma'(u) D_i u$, we can rearrange (5.6) to

$$
\int_\Omega \nu_i \eta w \gamma(u) D_i u = - \left( \int_\Omega \nu_i \gamma(u) D_i w + \int_\Omega \nu_i w \gamma(u) D_i \eta \right).
$$

Note that the function $\gamma'(u)$ is an indicator function for the set $\{|u| < \rho\}$. This, along with the fact that $|Du|^2/a = -\nu \cdot Du$, lets us write

$$
\int_{B^n_\rho(y) \cap \{|u-u(y)|<\rho\}} \frac{|Du|^2}{a} w \leq - \int_\Omega \nu_i \eta w \gamma'(u) D_i u
$$

$$
= \int_\Omega \nu_i \eta \gamma(u) D_i w + \int_\Omega \nu_i w \gamma(u) D_i \eta.
$$

Then we use Cauchy-Schwartz to see

$$
\int_\Omega \nu_i \eta \gamma(u) D_i w + \int_\Omega \nu_i w \gamma(u) D_i \eta \leq \int_\Omega |\nu| \eta \gamma(u) |Dw| + \int_\Omega |\nu w \gamma(u)| |D\eta|,
$$

after which we can use the facts that $|\nu| = 1$, $\gamma < 2\rho$ and $\gamma(u) = 0$ when $u < -\rho$ to see

$$
\int_\Omega |\nu \eta \gamma(u)| |Dw| + \int_\Omega |\nu w \gamma(u)| |D\eta| \leq 2\rho \int_{\{u>\rho\}} \eta |Dw| + 2\rho \int_{\{u>\rho\}} w |D\eta|.
$$
We can now use the conditions on \( \eta \) set out in the statement of the lemma to see that
\[
2\rho \int_{\{u>\rho\}} \eta |Dw| + 2\rho \int_{\{u>\rho\}} w |D\eta|
\]
\[
\leq 2\rho \int_{B_{2\rho} \cap \{u>\rho\}} \eta |Dw| + 2\rho \int_{B_{2\rho} \cap \{u>\rho\}} w |D\eta| \quad \text{by condition 4},
\]
\[
\leq 2\rho \int_{B_{2\rho} \cap \{u>\rho\}} \eta |Dw| + 4 \int_{B_{2\rho} \cap \{u>\rho\}} w \quad \text{by condition 2},
\]
\[
\leq 2\rho \int_{B_{2\rho} \cap \{u>\rho\}} \eta |Dw| + 4 \int_{B_{2\rho} \cap \{u>\rho\}} a \quad \text{since } \log a \leq a,
\]
and this completes the proof.

\[\square\]

**Lemma 5.11.** Let \( y \in \Omega \), and find \( \rho > 0 \) such that \( B_{2\rho}(y) \subset \Omega \), and let \( \eta \in C^1_c(\Omega) \) be a function satisfying the conditions in Lemma 5.10. Then the following inequality holds:
\[
\int_{B_{2\rho}(y) \cap \{u-u(y)>\rho\}} \eta |Dw| \leq \frac{C}{\rho} \int_{B_{2\rho}(y) \cap \{u-u(y)>-2\rho\}} a,
\]
where \( C = C(u) \) is independent of the choice of \( y \) and \( \rho \).

**Proof.** We will start with a relationship that we showed in (5.5) back in Lemma 5.6, namely
\[
\int_{g(u)} |\nabla w|^2 \varphi + \int_{g(u)} \nabla w \cdot \nabla \varphi \leq 0,
\]
which holds for any \( \varphi \in C^1_c(\Omega \times \mathbb{R}) \). If instead we take our test function as \( \varphi^2 \in C^1_c(\Omega \times \mathbb{R}) \), we can rewrite this as:
\[
0 \geq \int_{g(u)} |\nabla w|^2 \varphi^2 + \int_{g(u)} \nabla w \cdot \nabla \varphi^2
\]
\[
= \int_{g(u)} |\nabla w|^2 \varphi^2 + 2 \int_{g(u)} \varphi \nabla w \cdot \nabla \varphi.
\]
This gives
\[
\int_{g(u)} |\nabla w|^2 \varphi^2 \leq 2 \left( \int_{g(u)} \varphi^2 |\nabla w|^2 \right)^{1/2} \left( \int_{g(u)} |\nabla w|^2 \right)^{1/2},
\]
by Cauchy-Schwarz, and therefore
\[
\int_{g(u)} |\nabla w|^2 \varphi^2 \leq 4 \int_{g(u)} |\nabla \varphi|^2. \tag{5.7}
\]
We will also assume, without loss of generality, that \( y = 0 \), and \( u(y) = 0 \). Now, let us define the function that we will be using as our \( \varphi \). We set
\[
\varphi(x', x_{n+1}) = \eta(x') \lambda(x_{n+1}),
\]
where \( \eta \) is as in the statement of Lemma 5.10 and \( \lambda \) is a suitably differentiable cut off function that satisfies
1) \( \lambda(x) = 1 \) for all \( x \in (-\rho, \sup_{B_{2\rho}} u) \),

2) \( \lambda(x) = 0 \) for all \( x \in \mathbb{R}/(-2\rho, \rho + \sup_{B_{2\rho}} u) \),

3) \( |\lambda'(x)| \leq 2/\rho \) for all \( x \in \mathbb{R} \).

By choosing \( \lambda \), and therefore \( \varphi \), in this way, we are able to get a further bound for equation (5.7):

\[
\int_{\mathcal{G}(u)} |\nabla w|^2 \varphi^2 \leq 4 \int_{\mathcal{G}(u)} |\nabla \varphi|^2 \leq \frac{C^2}{\rho^2} \mathcal{H}^n(B),
\]

where

\[
B = \{ x \in \mathcal{G}(u) : |x'| < 2\rho \quad \text{and} \quad -2\rho < x_{n+1} < \rho + \sup_{B_{2\rho}} u \}
\]

is the set where \( \nabla \varphi \) is non-zero. Now, because \( w \) is independent of \( x_{n+1} \), we know that

\[
\nu_{n+1} |Dw| \leq |\nabla w|,
\]

from the definition of the tangential gradient, and recall that \( \nu_{n+1} = 1/a \). We now have all of the facts we need for our estimate:

\[
\int_{B_{2\rho} \cap \{u > -\rho\}} \eta |Dw| \leq \int_{B_{2\rho} \cap \{u > -\rho\}} a \eta |\nabla w|, \quad \text{by (5.9),}
\]

\[
\leq \int_B \eta |\nabla w| \, d\mathcal{H}^n,
\]

\[
\leq \left( \int_B \eta^2 |\nabla w|^2 \right)^{1/2} \mathcal{H}^n(B)^{1/2}, \quad \text{by Cauchy-Schwartz,}
\]

\[
\leq \left( \int_B \varphi^2 |\nabla w|^2 \right)^{1/2} \mathcal{H}^n(B)^{1/2}, \quad \text{since } \lambda = 1 \text{ on } B,
\]

\[
\leq \left( \int_{\mathcal{G}(u)} \varphi^2 |\nabla w|^2 \right)^{1/2} \mathcal{H}^n(B)^{1/2}, \quad \text{because } B \subset \mathcal{G}(u),
\]

\[
\leq \frac{C}{\rho} \mathcal{H}^n(B), \quad \text{by (5.8),}
\]

\[
= \frac{C}{\rho} \int_{B_{2\rho} \cap \{u > -2\rho\}} a,
\]

which completes the proof.

**Proof of Proposition 5.9.** As before, we have done most of the work now. We know that

\[
\int_{B_{2\rho}^n(\{u-u(y)\} < \rho)} |Dw| \, \frac{|Dw|^2}{a} \leq 4 \int_{B_{2\rho}^n(\{u-u(y)\} < \rho)} \frac{a}{4} + 2\rho \int_{B_{2\rho}^n(\{u-u(y)\} > \rho)} \eta |Dw|, \quad \text{from Lemma 5.10}
\]

\[
\leq 4 \int_{B_{2\rho}^n(\{u-u(y)\} > \rho)} a \, \frac{a}{2} + 2C_1 \int_{B_{2\rho}^n(\{u-u(y)\} > \rho)} a, \quad \text{by Lemma 5.11}
\]

\[
\leq C_2 \int_{B_{2\rho}^n(\{u-u(y)\} > 2\rho)} a,
\]

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and we are done. \hfill \Box

**Proposition 5.12.** For any \(y \in \Omega\), and \(\rho > 0\) such that \(B_{3\rho}^n(y) \subset \Omega\), we have the following inequality:

\[
\int_{B_{2\rho}^n \times \{u - u(y) > -2\rho\}} a \leq \omega_n (2\rho)^n + C \rho^n \left(1 + \frac{1}{\rho} \sup_{x \in B_{3\rho}^n(y)} [u(x) - u(y)]\right).
\]

**Proof.** As before, we assume without loss of generality that \(y = 0\) and \(u(y) = 0\). Now, by writing \(a = (1 + |Du|^2)/a\), we can see that

\[
\int_{B_{2\rho} \setminus \{u > -2\rho\}} a \leq \omega_n (2\rho)^n + \int_{B_{2\rho} \setminus \{u > -2\rho\}} \frac{|Du|^2}{a}.
\]

As usual, our task is to estimate the integral on the right hand and we do this by a careful choice of test function in the weak form of the minimal surface equation, from Lemma [5.3]. Let \(\mu\) be a function defined by

\[
\mu(t) = \begin{cases} 
  t + 2\rho, & \text{for } t \geq -2\rho, \\
  0, & \text{otherwise},
\end{cases}
\]

and let \(\phi\) be a suitably differentiable function which satisfies

1. \(0 \leq \phi \leq 1\) everywhere,
2. \(\phi = 1\) on \(B_{2\rho}\),
3. \(\phi = 0\) on \(\mathbb{R} / B_{3\rho}\),
4. \(|D\phi| \leq 2/\rho\) everywhere.

We then take \(\mu(u)\phi\) as our test function, so

\[
\int_{\Omega} \nu_i D_i (\mu(u)\phi) = 0,
\]

which, when we expand the derivative, gives

\[
\int_{\Omega} \nu_i \mu'(u) D_i u \phi = - \int_{\Omega} \nu_i \mu(u) D_i \phi.
\]

So, using the fact that \(|Du|^2/a = -\nu \cdot Du\) and our choice of test function, we know that

\[
\int_{B_{2\rho} \setminus \{u > -2\rho\}} \frac{|Du|^2}{a} \leq - \int_{\Omega} \nu_i \mu'(u) D_i u \phi,
\]

\[
= \int_{\Omega} \nu_i \mu(u) D_i \phi,
\]

\[
\leq \frac{2}{\rho} \int_{B_{3\rho} \setminus \{u \geq -2\rho\}} u + 2\rho, \quad \text{by Cauchy Schwartz and the definition of } \mu,
\]

\[
\leq C \rho^n \left[1 + \frac{1}{\rho} \sup_{B_{3\rho}} u\right],
\]

which completes the proof. \hfill \Box
We have now done most of the work we need in order to complete the proof of Theorem 5.1.

**Proof of Theorem 5.1.** We first collect the results we have proven in the propositions in this section to get a bound on $w = \log \sqrt{1 + |Du|^2}$. Let $y \in \Omega$, and choose $\rho > 0$ such that $B^n_{3\rho} \subset \Omega$. Then

\[
w(y) \leq \frac{1}{\omega_n \rho^n} \int_{G(w) \cap B^{n+1}_\rho(y)} w \, d\mathcal{H}^n \leq 1 + \frac{1}{\omega_n \rho^n} \int_{B_\rho(y)^n \times \{|u-u(y)|<\rho\}} |Du|^2 \frac{w}{a} \leq 1 + \frac{C_1}{\omega_n \rho^n} \int_{B^n_{2\rho}(y) \times \{u-u(y)>-2\rho\}} a \leq 1 + \frac{C_1}{\omega_n \rho^n} \left( \omega_n (2\rho)^n + C_2 \rho^n \left( 1 + \frac{1}{\rho} \sup_{x \in B^n_{3\rho}(y)} [u(x) - u(y)] \right) \right) \leq C_3 \left( 1 + \sup_{x \in B^n_{3\rho}(y)} \left[ \frac{u(x) - u(y)}{\rho} \right] \right).
\]

Now notice that we can take $\rho = d/3$, where $d = \text{dist}(y, \partial \Omega)$, to see that

\[
w(y) \leq C \left( 1 + \sup_{x \in B^n_d(y)} \left[ \frac{u(x) - u(y)}{d} \right] \right).
\]

By exponentiating both sides, we finally come to the conclusion that

\[
|Du(y)| \leq \exp w(y) \leq \exp \left( C \left( 1 + \sup_{x \in B^n_d(y)} \left[ \frac{u(x) - u(y)}{d} \right] \right) \right),
\]

which completes the proof. \qed

**References**


