# Existence and Continuation for Euler Equations

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22 March, 2014

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1 Introduction

1.1 Physical Background

Most equations arising from the study of continuum mechanics take the form of balance laws. A common (and reasonable) postulate is that of conservation of mass, which is expressed as

\[ \frac{\partial \rho}{\partial t} + \text{div}_x (\rho u) = 0 \]

for the mass density \( \rho = \rho(t, x) : I \times \Omega \rightarrow [0, \infty) \) and the velocity vector field \( u = u(t, x) : I \times \Omega \rightarrow \mathbb{R}^3 \), where \( I \) is some time interval and \( \Omega \subseteq \mathbb{R}^3 \) expresses the material body.

The Navier-Stokes equation is the standard equation expressing the balance of momentum. In its most general form it reads

\[ \rho \frac{\partial u}{\partial t} + u \cdot \nabla u = \text{div}_x T + f \]

where \( T : I \times \Omega \rightarrow \mathbb{R}^{3\times 3} \) is the stress tensor expressing surface forces within the fluid, and \( f : I \times \Omega \rightarrow \mathbb{R}^3 \) expresses the body force. The tensor \( T \) depends on \( u \) and its derivatives, and its form is a characteristic of the fluid; this form should always be specified in the study of the Navier-Stokes equation.

For this essay, we restrict our attention to the case of an incompressible fluid, i.e. we assume the fluid has constant density \( \rho \equiv \text{const} \), so mass conservation becomes

\[ \text{div}_x u = 0 . \]

Moreover, we also assume that the fluid is at constant temperature, so that we do not have to consider any heat generation or transport in an additional energy balance equation; and we assume that there are no body forces, i.e. \( f \equiv 0 \).

More importantly, we make the strong assumption that the only surface acting on the fluid is the pressure, i.e. \( T = -pI \). Then the Navier-Stokes equation reduces to the Euler equation,

\[ \rho \frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p . \]

This assumption is obtained by formally neglecting viscous stresses in the fluid. Such an approximation is known to be valid for fluid flow at high Reynolds numbers (i.e. high speed, low viscosity), away from any boundaries of the flow; indeed, most known fluids will form viscous boundary layers near boundaries at high-speed flow, where viscous stresses can no longer be neglected, even if the fluid is of very low viscosity. For simplicity we shall work in the whole space \( \Omega = \mathbb{R}^3 \) to avoid any boundary issues, and we shall work in non-dimensional variables with \( \rho \equiv 1 \).

In summary, we will concern ourselves with the Cauchy problem of the 3-dimensional incompressible Euler equation in whole space. These read

\[ \partial_t u + u \cdot \nabla u = -\nabla p \]

\[ \text{div} u = 0 \]

\[ u(t, \cdot) = u_0 \]

for the velocity field \( u \) and pressure field \( p \), which, for \( t \in \mathbb{R} \) sufficiently close to zero, should define functions \( u(t) : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) and \( p(t) : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \).

1.2 Notation and Conventions

In this essay we shall always define the Fourier transform of a Schwartz function \( f \in \mathcal{S}(\mathbb{R}^d) \) by

\[ \hat{f}(\xi) := \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f(x) \, dx , \]
so that the inverse transform is given by
$$\hat{f}(\xi) := \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f(x) \, dx.$$ 

We shall denote by $H^s(\mathbb{R}^d)$ the Sobolev space
$$H^s(\mathbb{R}^d) := \left\{ f \in S'(\mathbb{R}^d) \left\| \left(1 + |\xi|^2\right)^{s/2} \hat{f}(\xi) \in L^2(\mathbb{R}^d) \right\| \right\},$$
which is a Hilbert space equipped with the norm
$$\|f\|_{H^s} := \left( \int_{\mathbb{R}^d} \left|\hat{f}(\xi)\right|^2 (1 + |\xi|^2)^{s} \, d\xi \right)^{1/2}.$$
We remind the reader that when $s \in \mathbb{Z} \geq 0$, this norm is equivalent to the usual norm for Sobolev space of integer order, in terms of the sums of the $L^2$ norms of the distributional derivatives up to order $s$.

2 Solutions of the Euler Equation

We define the incompressible Euler equation to be the PDE given by
$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla \rho$$
$$\text{div} u = 0$$
$$u(x, 0) = u_0(x).$$

The main result we present here is the following.

**Theorem 2.1.** Given $u_0$ in $H^s(\mathbb{R}^3)$ with $\|u_0\|_{H^3} \leq N_0 < \infty$, there exists a time $T = T(N_0) > 0$ and a unique classical solution $u \in C([0, T]; H^3) \cap C^1([0, T]; H^2)$ to equation (1). Furthermore, if $T_1 = \inf\{T : \text{Equation (1) has no solution on } [0, T]\}$, then
$$\int_0^{T_1} \max_{x \in \mathbb{R}^3} |\nabla \times u(x, t)| \, dt = \infty$$

This existence result can also be proved from the abstract Kato theory but here we give a more classical approach. The continuation part of this theorem is special to 3 dimensions. We first consider the case of $u$ taking values in a bounded set and give results for dimensions 1, 2 and 3.

2.1 Equations of Compressible Fluid Flow

Throughout this section we work with an open set $G \subset \mathbb{R}^m$ such that $u : \mathbb{R}^N \to G$ for $N \in \{1, 2, 3\}$. Considering this set $G$ is natural since many physical quantities often can not take values in all of $\mathbb{R}^N$ for example density and total energy should be positive. We can also define $F_j : G \to \mathbb{R}^m$ to be a prescribed non-linear function and $S : G \times \mathbb{R}^N \times [0, \infty) \to \mathbb{R}^m$ to be a smooth function describing a source term. We can then consider the general equation
$$\frac{\partial u}{\partial t} + \sum_{j=1}^{N} \frac{\partial}{\partial x_j} F_j(u) = S(u, x, t)$$

(6)
We consider the compressible Euler equations,
\[
\begin{align*}
\frac{Dp}{Dt} + \gamma p \text{div} v &= 0 \\
\rho \frac{Dv}{Dt} + \nabla p &= 0 \\
\frac{DS}{Dt} &= 0
\end{align*}
\]  
(7)
for \( p = p^{1/\gamma} e^{-S_0/\gamma} \) and
\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \sum_{j=1}^{N} v_j \frac{\partial}{\partial x_j}.
\]

Equation (6) is more clearly seen to be a generalisation when written in a more classical form. [6] shows that this is the correct framework. Equation (1) is in some sense (see Theorem 2.14) the limit of equation (7) and equation (7) takes the form in (6) with \( \gamma = S_0 \). Since for us \( \gamma = S_0 \) is constant the last equation in (7). Thus we will require the following theorem. We can see that it is, in spirit, close to Theorem 2.1.

**Theorem 2.2** (Majda, [6]). Assume \( u_0 \in H^s \), \( s > N/2 + 1 \) and \( u_0 : \mathbb{R}^N \rightarrow G_1 \) for \( G_1 \subset \subset G \). Then there exists \( T > 0 \) such that equation (6) has a unique classical solution \( u \in C^1([0,T] \times [0,T], G_2) \) for \( G_2 \subset \subset G \).

Furthermore,
\[
\begin{align*}
u & \in C([0,T], H^s) \cap C^1([0,T], H^{s-1})
\end{align*}
\]  
(8)
and \( T \) only depends on \( \|u_0\|_s \), \( G_1 \).

We see that when \( N = 3 \), taking \( s = 3 \) in the assumptions will give us a similar result to Theorem (2.1) for bounded \( u \).

The plan is now to make rigorous, the following heuristic arguments:
1. Show uniform stability for the compressible solution to equation (7) independent of the “Mach number”.
2. Construct solution for equation (1) by taking the limit as the “Mach number” (defined in Definition 2.12) approaches zero.
3. Compressible Euler = Incompressible Euler + \( M(\text{Linear Acoustics}) + O(M^2) \) for a Mach number \( M \).

We can show uniqueness easily from the energy estimation methods discussed below or see [11].

**2.2 Existence in Theorem 2.2**

We split the proof into several parts. We first prove

**Theorem 2.3.** Under the hypotheses of Theorem 2.2, there exists a unique classical solution \( u \in C^1([0,T] \times \mathbb{R}^N) \) to equation (6) with
\[
u \in L^\infty([0,T], H^s) \cap C_w([0,T], H^s) \cap \text{Lip}([0,T], H^{s-1})
\]

Here we have used the notation \( C_w \) to be continuous functions on \([0,T]\) with values in the weak topology of \( H^s \). That is to say, for any \( \phi \in H^s \),
\[
(\phi, u(t)) = \sum_{|\alpha| \leq s} \int_{\mathbb{R}^N} D^\alpha \phi \cdot D^\alpha u(t) \, dx
\]
is a continuous function on \([0,T]\).

\( \text{Lip}([0,T], H^{s-1}) \) is the space of Lipschitz functions on \([0,T]\) with values considered in the norm topology of \( H^{s-1} \).

Once we have proved Theorem 2.2, we can deduce Theorem 2.2 from Theorem 2.4 below.
Theorem 2.4. Any classical solution of equation (6) $u$ taking values in $G_2 \subset G$ satisfying the regularity as in the conclusion of Theorem 2.2 on some interval $[0, T]$ also has the regularity property that $u \in C([0, T], H^s) \cap C^1([0, T], H^{s-1})$.

As in [6], we consider $S = 0$ for ease of exposition although the same proof will work for more general $S$. We are interested in constant $S = S_0$ and so this doesn’t take away understanding from our situation. The proof of Theorem 2.2 splits up into several key ideas:

1. Consider the linearised problem with mollified initial data.
2. Use an iteration scheme to construct a solution to the linear equation.
3. Show boundedness in the “high” norm.
4. Show contraction in the “low” norm.

The so called high and low norms come about since it is easier to find a contraction with a lower norm and combining this with the result for the high norm, we can get the required results in the norm that we are interested in - the high norm. This will become more transparent in the proof below.

When $S = 0$, any constant $u_0$ is a solution of (6). We can then consider solutions $u(x, t) = u_0 + v$ and thus obtain the linearised equations,

\[
\frac{\partial v}{\partial t} + \sum_{j=1}^{N} A_j(u_0) \frac{\partial v}{\partial x_j} = 0, \quad t > 0, \quad x \in \mathbb{R}^N
\]

\[
v(x, 0) = v_0(x)
\]

where $A_j(u) = \frac{\partial F_j}{\partial u}$, $j = 1, \ldots, N$ are $m \times m$ Jacobian matrices.

In many physical equations, for $u$ taking values in $G$, we can find a positive definite symmetric matrix $A_0(u)$ smoothly varying in $u$ with

1. $cI \leq A_0(u) \leq c^{-1}I$,
2. $A_0(u)A_j(u) = \tilde{A}_j(u)$, where $\tilde{A}_j(u) = \tilde{A}_j^*(u)$, for $j = 1, \ldots, N$.

We add the assumption that this $A_0$ exists for our proof. In the main case we are interested in, we can see that this holds with

\[
A_0(u) = \begin{pmatrix}
(\gamma p)^{-1} & 0 \\
0 & \rho(p, S) I_3
\end{pmatrix}
\]

One key result used in the proof Theorem 2.2 is the following stability lemma due to Friedrichs’ energy estimates [6].

Lemma 2.5. Suppose $v$ is a solution to the equations

\[
A_0(u) \frac{\partial v}{\partial t} + \sum_{j=1}^{N} \tilde{A}_j(u) \frac{\partial v}{\partial x_j} - B(u, x, t)v = f(t)
\]

\[
v(x, 0) = v_0(x)
\]

for $u$ a $C^1$ function taking values in $G_1 \subset G$ and $B$ a smoothly varying $m \times m$ matrix function. Then,

\[
\max_{t \in [0, T]} \|v\|_0 \leq c^{-1} \exp \left( \frac{1}{2} c^{-1} \|\text{div } A + B + B^*\|_{L^\infty} T \right) \left[ \|v(0)\|_0 + \int_0^T \|f(s)\|_0 ds \right]
\]

where $A = (A_0, \tilde{A}_1, \ldots, \tilde{A}_N)$.
This result follows from considering the energy defined as \( E(t) = (A_0(u), v) \) and using Gronwall’s inequality on \( \frac{\partial}{\partial t} E(t) \). This will also be useful later.

Finally, before the proof of Theorem 2.2 we present a result due to [8] which gives sharp inequalities for products of functions.

**Theorem 2.6 (Moser-type Inequalities).**

1. For \( f, g \in H^s, s > N/2 \) we have \( f \cdot g \in H^s \) and moreover \( \|f \cdot g\|_s \leq C_s \|f\|_s \|g\|_s \).

2. For \( f, g \in H^s \cap L^\infty \) and \( |\alpha| \leq s \),

\[
\|\partial^\alpha (fg)\|_0 \leq C_s(\|f\|_{L^\infty} \|\partial^\alpha g\|_0 + \|g\|_{L^\infty} \|\partial^\alpha f\|_0)
\]

3. For \( f \in H^s, Df \in L^\infty, g \in H^{s-1} \cap L^\infty \) and \( |\alpha| \leq s \),

\[
\|\partial^\alpha (fg) - fD^\alpha g\|_0 \leq C_s(\|Df\|_{L^\infty} \|\partial^{\alpha-1} g\|_0 + \|g\|_{L^\infty} \|\partial^\alpha f\|_0)
\]

4. Assume \( g(u) \) is a smooth vector-valued function on \( G \) and \( u \) is a continuous function with \( u(x) \in G_1 \) for all \( x \in \mathbb{R}^N \) and \( \overline{G_1} \subset \subset G \) and \( u \in H^s \cap L^\infty \). Then for \( s \geq 1 \),

\[
\|\partial^\alpha g(u)\|_0 \leq C_s\|\partial^\alpha g\|_{C^{s-1}(\overline{G_1})} \|u\|_{L^\infty}^{s-1} \|\partial^\alpha u\|_0
\]

The proof of these results can be found in [8] and partial proof is given in the section below (see Lemmas 3.9 and 3.14).

**Proof of Theorem 2.2.** As mentioned above we will split the proof into various parts.

**Step 1:** First of all we smooth the initial data using standard mollification tricks. We choose a function \( j \in C_0^\infty(\mathbb{R}^N) \) to be non-negative, supported in the unit ball and to have integral equal to 1. Then we define \( j_\varepsilon(x) = 1/\varepsilon^N j(x/\varepsilon) \) and set \( J_\varepsilon u(x) = j_\varepsilon * u(x) \). We also set

\[
\varepsilon_k = 2^{-k} \varepsilon_0
\]

\[
u_k(x) = J_{\varepsilon_k} u_0
\]

for \( \varepsilon_0 > 0 \) to be chosen later and \( k \in \mathbb{N} \).

**Step 2:** The key observation here is that in order to construct a smooth solution to equation (6), it suffices to differentiate the nonlinear term and apply the symmetrising matrix from above. Thus we consider the quasi-linear system (dropping the tilde notation)

\[
A_0(u) \frac{\partial u}{\partial t} + \sum_{j=1}^N A_j(u) \frac{\partial u}{\partial x_j} = 0
\]

\[
u(x, 0) = u_0(x)
\]

Now we can consider a classical iteration scheme.

Our initial estimate is \( u^{(0)}(x, t) = u_0^{(0)}(x) \) and we inductively define \( u^{(k+1)}(x, t) \) via the solution to the linear equation

\[
A_0(u^{(k)}) \frac{\partial u^{(k+1)}}{\partial t} + \sum_{j=1}^N A_j(u^{(k)}) \frac{\partial u^{(k+1)}}{\partial x_j} = 0
\]

\[
u^{(k+1)}(x, 0) = u_0^{(k+1)}(x)
\]

First of all, we need to show that these iterates are actually well defined.

Let \( G_2 \) be an open set such that \( \overline{G_1} \subset G_2 \) and \( \overline{G_2} \subset \subset G \). Since by assumption, \( s > N/2 \), we have

\[
\|v\|_{L^\infty(G_2)} \leq C \|v\|_s
\]

by a Sobolev embedding.
From the basic properties of mollification and the fact that \( u_0 \) takes values in \( G_1 \), it follows that there exists an \( R \) and \( \epsilon_0 \) such that if
\[
\|u - u_0^{(0)}\|_s \leq R, \text{ then } u(x) \in G_2 \text{ for all } x.
\]
and for the same \( \epsilon_0 \) we have
\[
\|u_0 - u_0^{(k)}\|_s \leq cR/4
\]
for all \( k = 0, 1, \ldots \), and with \( c \) from the equation (11) i.e. \( cI \leq A_0(u) \leq c^{-1}I \) for all \( u \in G_2 \).

Combining these facts we can inductively solve for \( u^{(k+1)} \) on some time interval \([0, T_k]\) and in fact \( u^{(k+1)} \in C^\infty([0, T_k] \times \mathbb{R}^N) \) where \( T_k \) is the largest time such that
\[
\|u^{(k)} - u_0^{(0)}\|_{s,T_k} \leq R
\]
holds where
\[
\|w\|_{s,T} \equiv \max_{t \in [0,T]} \|w\|_s
\]
for \( w \in L^\infty([0,T],H^s) \). In order for the problem (16) to have a solution on some interval of positive measure we need there to exist a time \( T_* > 0 \) such that \( T_k > T_* \) for all \( k \geq 1 \). Thus we need “boundedness in the high norm”. For now we will assume the following lemma.

**Lemma 2.7.** There exist constants \( L > 0 \) and \( T_* > 0 \) such that the solutions \( u^{(k+1)}(x,t) \) defined above satisfy
\[
\|u^{(k+1)} - u_0^{(0)}\|_{s,T_*} \leq R
\]
(22)
\[
\|\partial_t u^{(k+1)}\|_{s-1,T_*} \leq L
\]
(23)

We will defer the proof of Lemma 2.7 until later to keep the proof more transparent.

**Step 3:** The next task is to find a norm, a so called “low norm” such that we have convergence of \( u^{(k)} \) to a function \( u \). In fact we will look for a norm such that
\[
\sum_{k=1}^{\infty} \|u^{(k+1)} - u^{(k)}\| \leq \infty.
\]
This is not necessarily true in the high norm. Note that we need the norm to be strong enough so that the expressions in equation (16) converge to the desired limiting equation.

A naive choice of norm could be \( \| \cdot \|_{s,T_*} \) for \( T_* \leq T_* \) but if we use the energy estimate in (15) it can be shown that
\[
\|u^{(k+1)} - u^{(k)}\|_{s,T_*} \leq \alpha_k \|u^{(k)} - u^{(k-1)}\|_{s,T_*}.
\]
This is almost the contraction that we would like but unfortunately \( \alpha_k \) depends on \( \|u^k\|_{s+1} \) and this is not bounded. It will turn out that it is sufficient to work in the norm \( \| \cdot \|_{0,T_*} \). In fact we have the following lemma for contraction in the low norm

**Lemma 2.8.** There exists a time \( T_* \leq T_* \), \( \alpha < 1 \) and \( \{\beta_k\}_{k=1}^\infty \) with \( \sum |\beta_k| < \infty \) such that
\[
\|u^{(k+1)} - u^{(k)}\|_{0,T_*} \leq \alpha \|u^{(k)} - u^{(k-1)}\|_{0,T_*} + \beta_k
\]
for all \( k \geq 1 \).

This lemma follows via a simple application of the energy estimate (15) to
\[
A_0(u^{(k)}) \frac{\partial}{\partial t} (u^{(k+1)} - u^{(k)}) + \sum_{j=1}^{N} A_j(u^{(k)}) \frac{\partial}{\partial x_j} (u^{(k+1)} - u^{(k)}) = - \sum_{j=1}^{N} (A_j(u^{(k)}) - A_j(u^{(k-1)})) \frac{\partial u^{(k+1)}}{\partial x_j}
\]
and then using Lemma 2.7 and Taylor’s theorem.

**Step 4:** We now show the convergence of the iteration equation (17).
Note that for any norm, if \( \|u^{(k+1)} - u^{(k)}\| \leq \alpha \|u^{(k)} - u^{(k-1)}\| + \beta_k \) with \( \alpha < 1 \) and \( \sum_{k=1}^{\infty} |\beta_k| < \infty \), then
\[
\sum_{k=1}^{\infty} \|u^{(k+1)} - u^{(k)}\| < \infty.
\]
Thus by Lemma 2.8 there exists \( u \in C([0,T_*], L^2(\mathbb{R}^N)) \) such that \( u^{(k)} \to u \) in the \( \| \cdot \|_{0,T_*} \) norm.

We also know from Lemma 2.7 that
\[
\|u^{(k)}\|_{s,T_*} + \| \frac{\partial u^{(k)}}{\partial t} \|_{s-1,T_*} \leq C
\]
(24)
\[
u^{(k)}(x,t) \in G_2 \subset \subset G, \text{ for all } (x,t) \in \mathbb{R}^N \times [0,T_*]
\]
(25)
Here, and everywhere else, \( C \) is an arbitrary constant.

One more result that we require follows from Sobolev interpolation inequalities with fractional Sobolev spaces [1, p.135]. That is,
\[
\|v\|_{s'} \leq C_s \|v\|_{0}^{1-s'/s} \|v\|_{s}^{s'/s}
\]
(26)
for all \( 0 \leq s' \leq s \).

Putting these two facts together gives us the result that
\[
\|u^{(k)} - u^{(l)}\|_{s',T_*} \leq C \|u^{(k)} - u^{(l)}\|_{0,T_*}^{1-s'/s}.
\]
(27)
as the limit is uniform in time. Hence, we have convergence in the \( s' \) norm. More specifically,
\[
\lim_{k \to \infty} \|u^{(k)} - u\|_{s',T_*} \to 0
\]
(28)
for all \( s' < s \). Thus, if we also choose \( s' > N/2 + 1 \), we have that
\[
u^{(k)} \to u, \text{ in } C([0,T_*], C^1(\mathbb{R}^N)).
\]
(29)
We see that we also get convergence of the time derivative of \( u^{(k)} \). In particular, from equation (17) we have
\[
\frac{\partial u^{(k+1)}}{\partial t} - A_0 u^{(k)} = \sum_{j=1}^{N} A_j(u^{(k)}) \frac{\partial u^{(k+1)}}{\partial x_j}
\]
and so
\[
\frac{\partial u^{(k)}}{\partial t} \to \frac{\partial u}{\partial t}, \text{ in } C([0,T_*] \times \mathbb{R}^N)
\]
The proof that
\[
u \in C_w([0,T], H^s) \cap \text{Lip}([0,T_*], H^{s+1})
\]
follows from strong convergence in \( C([0,T_*], H^s) \) as shown above. For more details see [6] but here we skip over this for brevity since the techniques are fairly standard. The proof is now complete, up to the proof of Lemma 2.7.

Step 5: For the proof of Lemma 2.7 we proceed by induction For \( k = 0, T_0 = \infty \) and so the base case is easy. We set \( v^{(k+1)} = u^{(k+1)} - u^{(0)} \). Then we have
\[
A_0(u^{(k)}) \frac{\partial v^{(k+1)}}{\partial t} + \sum_{j=1}^{N} A_j(u^{(k)}) \frac{\partial v^{(k+1)}}{\partial x_j} = - \sum_{j=1}^{N} A_j(u^{(k)}) \frac{\partial}{\partial x_j} u^{(0)} = h^k
\]
(30)
u^{(k+1)}(x,0) = u^{(k+1)}(x) - u^{(0)}(x)

By the induction hypothesis, \( u^{(k)} \in G_2 \) for all \( (x,t) \in \mathbb{R}^N \times [0,T_*] \) for some \( T_* \) to be chosen later. We now ignore the superscripts above for improved readability. Consider \( u, v \in C^\infty \) on
$\mathbb{R}^N \times [0, T_*]$ with $u$ taking values in $G_2$.

$$A_0(u)\frac{\partial v}{\partial t} + \sum_{j=1}^{N} A_j(u)\frac{\partial v}{\partial x_j} = h$$

$$v(x, 0) = v_0(x)$$

Also, we use the shorthand notation $v_\alpha = D^\alpha v$ for $|\alpha| \leq s$. We can then obtain a similar equation to the above plus some error terms and when $\alpha = 0$ we get the same result as above. We also omit the $u$ to improve presentation.

$$A_0\frac{\partial v_\alpha}{\partial t} + \sum_{j=1}^{N} A_j\frac{\partial v_\alpha}{\partial x_j} = A_0D^\alpha(A_0^{-1}h) + F_\alpha$$

$$v_\alpha(x, 0) = D^\alpha v_0(x)$$

where $F_\alpha$ represents the commutator terms given by

$$F_\alpha = \sum_{j=1}^{N} A_0 \left[ (A_0^{-1}A_j)\frac{\partial v_\alpha}{\partial x_j} - D^\alpha(A_0^{-1}A_j)\frac{\partial v}{\partial x_j} \right]$$

We are now in a position to proceed by induction. We assume that the result of Lemma 2.7 holds for $u^{(k)}$ and for some $L$ and $T_*$ and estimate $\|u^{(k+1)} - u_0^{(0)}\|_{s,T_*}$. We will then be done, since $R$ determines a choice of $L$, independently of $T_*$, by simply studying the equation for $\frac{\partial u^{(k+1)}}{\partial t}$ and using Moser-type inequalities (see Theorem 2.6 above). In order to obtain a bound on $\|u^{(k+1)} - u_0^{(0)}\|_{s,T}$ for some $T$, we must control the right hand side of equation (32) we can use the energy estimates in (15) and see that we will be done if we can show that for some $C$ depending on $G_2$, $\|u\|_s$, $R$ and $s$ only, we have

$$\sum_{1 \leq |\alpha| \leq s} \|F_\alpha\|^2 + \sum_{|\alpha| \leq s} \|A_0D^\alpha(A_0^{-1}h)\|^2 \leq C(\|v\|^2_s + 1)$$

(34)

If we temporarily assume this bound, we will be done. Indeed, applying the energy estimate to $v_\alpha^{(k+1)}$ and summing over all $\alpha$, we see that

$$\|u^{(k+1)} - u_0^{(0)}\|_{s,T} \leq c^{-1} e^{(LC+C)T}(\|u^{(k+1)} - u_0^{(0)}\|_s + TC)$$

(35)

with $c$ from equation (11) and

$$\|u^{(k+1)} - u_0^{(0)}\|_s \leq cR/2.$$

Thus, we get for example

$$\|u^{(k+1)} - u_0^{(0)}\|_{s,T} \leq e^{(L+1)CT}(R/2 + TC)$$

and from here, it is easy to find a fixed $T_*$ such that

$$\|u^{(k+1)} - u_0^{(0)}\|_{s,T} \leq R$$

as required for the inductive step.

**Step 6:** Now all that remains, is to show the estimate in equation (34). We prove this bound using Theorem 2.6.

Now we improve this result to show Theorem 2.2.

**Proof of Theorem 2.4.** If $u(x, t)$ is the local solution from Theorem 2.2 we see from the proof above that $A_0(u(x, t)) \in C([0, T_*], C(\mathbb{R}^N))$ and $cT \leq A_0(u(x, t)) \leq c^{-1} I$ for $(x, t) \in \mathbb{R}^N \times [0, T_*]$. We write $A_0(t) \equiv A_0(u(x, t))$ in this case. For $t \in [0, T_*]$, we define the norm for $v \in H^s$

$$\|v\|^2_{s,A_0(t)} \equiv \sum_{|\alpha| \leq s} \int_{\mathbb{R}^N} (D^\alpha v, A_0(t)D^\alpha v) \, dx.$$
From above, we have equivalence of norms, i.e.
\[ c\|v\|^2 \leq \|v\|_{s,A_0(t)}^2 \leq c^{-1}\|v\|^2, \quad (36) \]

It is also straightforward to show that
\[ \limsup_{t \to 0} \|v(t)\|_{s,A_0(t)}^2 = \limsup_{t \to 0} \|v(t)\|_{s,A_0(0)}^2 \quad \text{(37)} \]

for all \( v(t) \in C_w([0,T_*],H^s) \).

With these observations in mind, we can simplify our task by reducing the proof to showing the result in a simpler setting.

1. It suffices to show that \( u \in C([0,T],H^s) \) from the explicit form of equation (16) since then trivially \( u \in C^1([0,T],H^{s-1}) \).

2. We will only show strong right continuity at \( t = 0 \) as the same argument gives us strong right continuity at every \( 0 \leq t < T \) and equation (16) and the argument below are reversible in time, and thus strong right continuity on \([0,T]\) is equivalent to strong left continuity on \((0,T]\).

Recall the general result that if \( \{w_n\} \) is a weakly convergent sequence to \( w \) in a Hilbert space, then \( w_n \) converges strongly to \( w \) if and only if \( \|w\| \geq \limsup_{n \to \infty} \|w_n\| \). Applying this to \( H^s \) with norm \( \|\cdot\|_{s,A_0(0)} \), we see that strong right continuity at \( t = 0 \) in \( H^s \) is given if we have
\[ \|v\|_{s,A_0(t)}^2 \geq \limsup_{t \to 0} \|v\|_{s,A_0(t)}^2 \]

This will be proved using equation (37) and “taking \( \limsup \) as \( t \) tends to \( 0 \) in following Lemma”.

**Lemma 2.9.** Let \( u \) be a local solution from Theorem 2.2 on some interval \([0,T_*]\). Then there exists a function \( f \in L^1([0,T_*]) \) such that
\[ \|u(t)\|_{s,A_0(t)}^2 \leq \|u_0\|_{s,A_0(0)}^2 + \int_0^t |f(s)| \, ds \]
for all \( t \in (0,T_*] \).

**Proof of Lemma 2.9.** Recall that \( u^k \in C^\infty \cap H^s \). By the energy techniques mentioned before, it is possible to show that for \( 0 \leq t \leq T_* \) we have
\[ \frac{\partial}{\partial t} \sum_{|\alpha| \leq s} \int_{\mathbb{R}^N} (D^\alpha u^{(k+1)}, A_0(u^{(k)})D^\alpha u^{(k+1)}) = \int_{\mathbb{R}^N} (\text{div} A(u^{(k)})u^{(k+1)}, u^{(k+1)}) + 2 \int_{\mathbb{R}^N} F_s^{k+1}(u^{(k+1)}).
\]

Here we have defined \( F_s^{k+1} \) in the following way, with the notation \( u^{(k+1)}_n \equiv D^\alpha u^{(k+1)} \),
\[ F_s^{k+1} = \sum_{1 \leq |\alpha| \leq s} \sum_{1 \leq j \leq N} A_0(u^{(k)}) A_0(u^{(k)})^{-1} A_j(u^{(k)}) \frac{\partial u^{(k+1)}_n}{\partial x_j} - D^\alpha (A_0^{-1} A_j(u^{(k)}) \frac{\partial u^{(k+1)}_n}{\partial x_j}) \]  

By Theorem 2.6 and Lemma 2.7, we can bound the right hand side of equation (38) by some absolutely integrable function \( f \) on \([0,T_*]\). We now integrate both sides of equation (38) between \( 0 \) and \( t \) and rearrange to see that
\[ \sum_{|\alpha| \leq s} \int_{\mathbb{R}^N} (D^\alpha u^{(k+1)}(t), A_0(u^{(k)}(t))D^\alpha u^{(k+1)}(t)) = \sum_{|\alpha| \leq s} \int_{\mathbb{R}^N} (D^\alpha u^{(k+1)}_0, A_0(u^{(k)}_0)D^\alpha u^{(k+1)}_0) + \int_0^t |f(s)| \, ds \]
Since \( \{u_0^{(k)}\} \) is defined via a mollification procedure, it is simple to see from standard convergence results that
\[
\lim_{k \to \infty} \sum_{|\alpha| \leq s} \int_{\mathbb{R}^N} (D^\alpha u_0^{(k)}(x), A_0 u_0^{(k)}(x)) = \|u_0\|_{s,A_0(0)}^2.
\]
(40)

From equation (28) we see that
\[
\lim_{k \to \infty} \max_{t \in [0,T^*]} \|A_0(u^{(k)}(t)) - A_0(u(t))\|_{L^\infty} = 0.
\]
(41)

We also have weak convergence as a result of Theorem 2.2 in the sense that, for a fixed \( t > 0 \), we have
\[
\lim_{k \to \infty} \sup_{t \in [0,T]} (u^{(k+1)}(t), u^{(k)}(t))_{s,A_0(t)} \geq (u(t), u(t))_{s,A_0(t)}.
\]
(42)

We can now conclude the proof of Lemma 2.9 and thus the proof of Theorem 2.4 by noting that by (36) and (37) we have that
\[
\|u(t)\|_{s,A(t)}^2 \leq \limsup_{k \to \infty} \sum_{|\alpha| \leq s} \int_{\mathbb{R}^N} (D^\alpha u^{(k+1)}(x), A_0(u^{(k)}(x))) D^\alpha u^{(k+1)}(x).
\]

We now state useful corollaries of Theorem (2.2) and the sharp continuation principle in [6, p.46]. We do not give the proof here.

**Proposition 2.10.** Assume \( u_0 \in H^s \) for some \( s > N/2 + 1 \) and that \( u \) is a classical solution of equation (6) on some interval \([0,T]\) with \( u \in C^1([0,T] \times \mathbb{R}^N) \). Then \( u \in C([0,T],H^s) \). In particular, if \( u_0 \in \cap_s H^s \) on any interval \([0,T]\) where \( u \in C([0,T],H^s) \) for some \( s_0 > N/2 + 1 \), then \( u \in C^\infty([0,T] \times \mathbb{R}^N) \).

The next corollary is a continuation principle which characterises the behaviour of the solution as we approach the endpoint of the maximum interval of the solution to our equation.

**Proposition 2.11.** Assume \( u_0 \in H^s \) for some \( s > N/2 + 1 \). Then \([0,T]\) with \( T < \infty \) is the maximal interval of \( H^s \) existence if and only if either

1. \( \|u_t\|_{L^\infty} + \|Du\|_{L^\infty} \rightarrow T \), or
2. \( u(x,t) \) escapes every compact subset \( K \subset G \), as \( t \rightarrow T \).

For a special case of this, see Theorem 3.12.

### 2.3 Constructive Existence for the Incompressible Euler Equations

Notice that the above framework can not be directly applied to the incompressible Euler equations. For this reason we consider the incompressible equations as the limiting case of the compressible equations that we just studied. More specifically, we are now ready to prove the following result which states that the solutions for the compressible Euler equations converge as the Mach number tends to zero and the resulting function satisfies the incompressible Euler equations. That is, we want to show that the equations of isentropic compressible fluid flow
\[
\frac{Dp}{Dt} + \rho \text{div} v = 0
\]
(43)
\[
\frac{Dv}{Dt} + \frac{1}{\rho} \nabla p = 0
\]
(44)
\[
p = A\rho^\gamma, \quad \gamma > 1.
\]
(45)
converge as the “Mach number (defined below in Definition 2.12) tends to zero” to the incompressible Euler equations given by

\[
\rho_0 \frac{Dv^\infty}{Dt} + \nabla p^\infty = 0 \tag{46}
\]

\[
\text{div} v^\infty = 0. \tag{47}
\]

The advantage of this approach is that we obtain a constructive existence of a solution. First of all we “non-dimensionalise” the compressible equations. We consider the initial data given by

\[
\rho(x,0) = \rho_0(x), \ v(x,0) = v_0(x) \tag{48}
\]

and we set \( \rho_{\text{max}} = \max \rho_0(x) \) and \( |v_{\text{max}}| = \max |v_0(x)| \). We then normalise and change variables according to

\[
\tilde{\rho} \equiv \frac{\rho}{\rho_{\text{max}}}, \quad \tilde{v} = \frac{v}{|v_{\text{max}}|}.
\]

We then obtain the non-dimensional form of the compressible Euler equations

\[
\frac{D\tilde{\rho}}{Dt} + \tilde{\rho} \text{div} \tilde{v} = 0 \tag{51}
\]

\[
\frac{D\tilde{v}}{Dt} + \lambda^2 \nabla p(\tilde{\rho}) = 0 \tag{52}
\]

where we write

\[
\lambda^2 = (\frac{dp}{d\rho}(\rho_m) \frac{1}{|v_{\text{max}}|^2}(\gamma A)^{-1}
\]

which has no dimension.

**Definition 2.12.** Using the notation above, the Mach number \( M \) is defined to be the ratio of typical fluid speed \(|v_{\text{max}}|\) to the typical sound speed \( c(\rho_{\text{max}}) \equiv \left( \frac{dp}{d\rho} \right)^{-\frac{1}{2}} \). i.e

\[
M = \frac{|v_{\text{max}}|}{c(\rho_{\text{max}})} \tag{53}
\]

and so \( \lambda = M^{-1}(\gamma A)^{-\frac{1}{2}} \).

We now change notation again for convenience by removing the primes and tildes and write the equation in terms of \((p^\lambda, v^\lambda)\) satisfying

\[
(\gamma p^\lambda)^{-1} \frac{Dp^\lambda}{Dt} + \text{div} v^\lambda = 0 \tag{54}
\]

\[
\frac{Dv^\lambda}{Dt} + \lambda^2 \nabla p^\lambda = 0 \tag{55}
\]

where \( \rho = (p^\lambda)^{1/\gamma} e^{-a/\gamma} \) and \( a \) is a constant source term. The initial conditions are given by

\[
p^\lambda(x,0) = p_0^\lambda(x), \ v^\lambda(x,0) = v_0^\lambda(x). \tag{56}
\]

If we proceed in a formal manner by expanding, for example \( p^\lambda = p_0 + \lambda^{-1} p_1 + \lambda^{-2} p_2 + O(\lambda^{-3}) \) and similarly for \( v^\lambda \) and use the orthogonal decomposition of \( L^2(\mathbb{R}^N) \) into divergence free vector fields and gradient fields (see Hodge decomposition in Lemma 3.2), we obtain hints to the the required convergence but it turns out that we do not actually obtain the correct result. For a more detailed overview of the physical interpretation and formal derivation of the convergence, see [6, Section 2.4]. Here, for brevity of exposition, we go straight into important stability results for convergence.

It turns out that the correct asymptotic expansion of \((p^\lambda, v^\lambda)\) is given by

\[
p^\lambda = P_0 + \lambda^{-2} (p^\infty(x,t) + p_1(x,t,\lambda)) + O(\lambda^{-3}) \tag{57}
\]

\[
v^\lambda = v^\infty + \lambda^{-1} (v_1(x,t,\lambda)) + O(\lambda^{-2}). \tag{58}
\]
Theorem 2.13. Assume the initial data \((p_0^\lambda(x), v_0^\lambda(x))\) satisfies the stability estimate
\[
\|\lambda(p_0^\lambda(x) - P_0)\|_{s_0} + \|v_0^\lambda(x)\|_{s_0} \leq R
\] for some fixed \(R > 0\), \(\lambda \geq 1\) and \(s_0 = \lfloor N/2 \rfloor + 2\). Then there is a fixed time interval \([0, T_0]\) with \(T_0 > 0\) and \(\lambda_0(R)\) such that for all \(\lambda \geq \lambda_0(R)\) the compressible Euler equations have a classical solution on \([0, T_0]\) and
\[
(\lambda(p^\lambda - P_0), v^\lambda) \in C([0, T_0], H^{s_0}) \cap C^1([0, T_0], H^{s_0-1})
\]
and
\[
\|\lambda(p^\lambda - P_0)\|_{s_0, T_0} + \|v^\lambda\|_{s_0, T_0} \leq R'
\] for some fixed constant \(R' > 0\).

We are now in a position to prove the main result of this section.

Theorem 2.14. Consider the solutions of the system
\[
\gamma^{-1}(p^\lambda) \frac{Dp^\lambda}{Dt} + \text{div}v^\lambda = 0
\]
\[
\rho(p^\lambda) \frac{Dv^\lambda}{Dt} + \lambda^2 \nabla p^\lambda = 0
\]
with initial data given by
\[
v^\lambda(x, 0) = v^\infty_0 + \lambda^{-1} v^1_0(x), \quad \text{div}v^\infty_0 = 0
\]
\[
p^\lambda(x, 0) = P_0 + \lambda^{-2} p^1_0(x)
\]
belonging to \(H^{s_0}\) and satisfying the stability estimate (59). From Theorem 2.13 we can find a fixed time interval \([0, T_0]\) where a classical solution exists. Then as \(\lambda \to \infty\) there exists \(v^\infty \in L^\infty([0, T_0], H^{s_0})\) such that
\[
v^\lambda \to v^\infty \quad \text{in} \ C([0, T_0], H^{s_0-\varepsilon})
\]
for any \(\varepsilon > 0\).

In fact,
\[
v^\infty \in C([0, T_0], H^{s_0}) \cap C^1([0, T_0], H^{s_0-1})
\]
and is a classical solution of the incompressible Euler equations. In other words, there exists a \(p^\infty\) such that
\[
\rho_0 \frac{Dv^\infty}{Dt} = -\nabla p^\infty
\]
\[
\text{div}v^\infty = 0
\]
\[
v^\infty(x, 0) = v^\infty_0(x).
\]
where \(\rho_0 = \rho(P_0)\). Moreover, the mean pressure \(p^\infty\) is the limit of \(p^\lambda\) in the weak-* sense. More specifically,
\[
\nabla p^\lambda \to \nabla p^\infty \quad \text{weak-* in} \ L^\infty([0, T_0], H^{s_0-1}).
\]

We give a sketch of the proof here. The proof can be found in full detail in [6].

Proof. As before we wish to show results in a low norm and use estimates in a high norm to recover the regularity. Here we show compactness in a low norm. It is convenient to write this in the abstract setting and so we use the notation below for the time being. We wish to look at
\[
A_0^\lambda \frac{Du^\lambda}{Dt} + \sum_{j=1}^N A_j^\lambda \frac{Du^\lambda}{Dx_j} = 0, \quad u^\lambda(x, 0) = u_0^\lambda(x),
\]
where \(w^\lambda = (\lambda(p^\lambda - P_0), v^\lambda)^T\) and
\[
A^\lambda_j = A_j(\lambda^{-1}u^\lambda, u^\lambda) + \lambda A_j^0, \quad 1 \leq j \leq N \\
A^\lambda_0 = A_0(\lambda^{-1}u^\lambda)
\]
for \(A^0_j\) constant symmetric matrices and \(cI \leq A_0(v) \leq c^{-1}I\) for \(|v|\) bounded and for these \(v\), \(A_j(v, y)\) are smoothly varying for arbitrary \(y \in \mathbb{R}^m\) [6].

From Theorem 2.13 we also have the bound
\[
\lambda^{-1}u^\lambda \frac{\partial A_0}{\partial u} |_{\lambda^{-1}u^\lambda} + \sum_{j=1}^N (\frac{\partial}{\partial u} A_j^0(\lambda^{-1}u^\lambda, u^\lambda))u_{x_j} \leq 0
\]
and
\[
B^\lambda = \lambda^{-1}u^\lambda \frac{\partial A_0}{\partial u} |_{\lambda^{-1}u^\lambda} + \sum_{j=1}^N (\frac{\partial}{\partial u} A_j^0(\lambda^{-1}u^\lambda, u^\lambda))u_{x_j}.
\]

From the properties of the matrices in (70) and Theorem 2.13 we see that we can uniformly bound \(B^\lambda\). In particular,
\[
\|B^\lambda(t)\|_{L^\infty} \leq C, \quad 0 \leq t \leq T_0.
\]
This means that we can use the energy estimate (15) again applied to \(w^\lambda\) and see that
\[
\|u^\lambda_t\|_{0, T_0} \leq C\|u^\lambda_t(0)\|_0, \quad \lambda \geq \lambda_0(R).
\]
Thus we can bound \(\|u^\lambda_t\|_{0, T_0}\) by \(R'\) assuming that we can bound \(\|u^\lambda_t(0)\|_0\) independently of \(\lambda\).

By the smoothness of \(A^\lambda_j\) and since \(\lambda\) is bounded below, we can see that, from equation (72), a sufficient condition for this is that
\[
\left\| \lambda \sum_{j=1}^N A_j^0 \frac{\partial}{\partial x_j} u^\lambda_0(x) \right\|_0 \leq \tilde{R}
\]
for some \(\tilde{R} > 0\).

In our case, the initial conditions given in the theorem satisfy this. Thus going back to our specific case we have the estimate,
\[
\lambda\|p^\lambda\|_{0, T_0} + \|v^\lambda_t\|_{0, T_0} \leq R'.
\]
From Theorem 2.13 we also have the bound
\[
\lambda\|p^\lambda - P_0\|_{s_0, T_0} + \|v^\lambda\|_{s_0, T_0} \leq R'.
\]
By using the interpolation inequality in (26), with \(s' = s_0 - \varepsilon\) and the two bounds above give us equicontinuity in \(H^{s_0-\varepsilon}_{\text{loc}}\) and uniform boundedness is clear. Therefore, by Arzela-Ascoli, we can choose a convergent subsequence such that
\[
v^\lambda \rightarrow v^\infty, \quad \text{in } C([0, T_0], H^{s_0-\varepsilon}_{\text{loc}})
\]
for some \(v^\infty \in L^\infty([0, T_0], H^{s_0})\) from the uniform bound on the sequence above. In a similar way we also have \(v^\infty \in \text{Lip}([0, T_0], L^2) \cap C([0, T], C^1(\mathbb{R}^N))\).

Now that we have a candidate solution to the incompressible Euler equations we can show that it is indeed a solution. By the equation (62) and the above estimates we see that
\[
\lambda\|\text{div} v^\lambda\|_{0, T_0} \leq \lambda\|\gamma^{-1}(p^\lambda)^{-1} \frac{Dp^\lambda}{Dt}\|_{0, T_0} \leq C
\]
for $C \equiv C(R')$. Therefore from the above considerations we can take limits as $\lambda \to \infty$ to see that

$$\text{div}^\infty = 0$$

(81)

We now show that $v^\infty$ satisfies a weak form of equation (66). To this end we take $w$ to be an arbitrary vector valued function with $w \in H^s \cap S$ for $S$ the Schwartz space on $\mathbb{R}^N$. Suppose also that $\text{div} w = 0$. We also define $\phi \in C_0^\infty((0,T_0))$ to be a smooth real valued test function. We write $(\cdot, \cdot)_0$ for the $L^2$ inner product.

By the orthogonal decomposition of $L^2$ into divergence free vector fields and gradient fields we see that

$$(w, \lambda^2 \nabla p^\lambda)_0 = 0$$

Therefore when we take the inner product of $w$ with the second equation in (62), multiply by the test function $\phi$ and then integrate over $t$ and use integration by parts to obtain

$$0 = \int_0^{T_0} -\frac{\partial \phi}{\partial t}(w, \rho^\lambda v^\lambda)_0 + \phi(t)(w, \rho^\lambda v^\lambda \cdot \nabla v^\lambda)_0 - \lambda^{-1} \phi(t)(w, \frac{\partial \rho}{\partial p}(\rho^\lambda) \frac{\partial \tilde{p}^\lambda}{\partial t} v^\lambda)_0 \, dt$$

(82)

where $\rho^\lambda = \rho(p^\lambda)$ and as before $\tilde{p}^\lambda = \lambda(p^\lambda - P_0)$.

Therefore by (78) and (79) we have

$$\rho^\lambda \to \rho(P_0) = \rho_0, \text{ uniformly on } [0,T_0] \times \mathbb{R}^N$$

and

$$\left\| \frac{\partial \tilde{p}^\lambda}{\partial t} v^\lambda \right\|_{0,T_0} \leq C(R'), \text{ (for } \lambda \text{ sufficiently large)}$$

respectively.

From the result in (80) and dominated convergence theorem, since $w \in S$ we also have convergence in the first two integrands above. More specifically,

$$(w, \rho^\lambda v^\lambda \cdot \nabla v^\lambda)_0 \to (w, \rho_0 v^\infty \cdot v^\infty)_0$$

(83)

$$(w, \rho^\lambda)_0 \to (w, \rho_0 v^\infty)_0$$

(84)

uniformly on $[0,T_0] \times \mathbb{R}^N$. So we can pass the limit in equation (82) as $\lambda \to \infty$ to see that

$$0 = \int_0^{T_0} \left( -\frac{\partial \phi}{\partial t}(w, \rho_0 v^\infty)_0 + \phi(w, \rho_0 v^\infty \cdot \nabla v^\infty)_0 \right) \, dt$$

(85)

for all $\phi \in C_0^\infty((0,T_0))$ and $w \in H^s \cap S$ with $\text{div} w = 0$. These $w$ are dense in the space of $L^2$ functions with these properties. Thus we have shown $v^\infty$ to be a weak solution to equation (66) with $v^\infty(x,0) = v_0^\infty(x)$ and

$$v^\infty \in \text{Lip}([0,T_0],L^2) \cap C([0,T_0],C^1) \cap L^\infty([0,T_0],H^{s_0}).$$

$s_0 = [N/2] + 2$ and from above,

$$\frac{1}{\rho_0} \mathcal{P}(v^\infty \cdot \nabla v^\infty) \in L^\infty([0,T_0],H^{s_0-1})$$

where $\mathcal{P}$ is the orthogonal projection from $L^2$ onto the space of divergence-free vector fields. The weak equation above tells us that

$$\frac{\partial v^\infty}{\partial t} = -\mathcal{P}(v^\infty \cdot \nabla v^\infty)$$

in the sense of distributions. Therefore, $v^\infty \in \text{Lip}([0,T_0],H^{s_0-1}) \cap L^\infty([0,T_0],H^{s_0})$. Combining everything together we note that also $v^\infty \in C([0,T_0],C^1(\mathbb{R}^N)) \cap \text{Lip}([0,T_0],H^{s_0-1}) \cap L^\infty([0,T_0],H^{s_0}) \cap C([0,T_0],H^{s'})$ for $s' < s$. Finally, we can show that we also have $\frac{2v^\infty}{\rho_0} \in$
C([0, T_0] \times \mathbb{R}^N) and so \( v^\infty \in C^1([0, T_0] \times \mathbb{R}^N) \) and is therefore a classical solution of (66). The fact that \( v^\infty \in C([0, T_0], H^{s_0}) \cap C^1([0, T_0], H^{s_0-1}) \) follows by a similar argument to the proof of Theorem 2.4. The pressure term can also be shown to converge (see [9]). The idea is that we know that
\[
P(\frac{\partial v^\infty}{\partial t} + v^\infty \cdot \nabla v^\infty) = 0
\]
and so
\[
\frac{\partial v^\infty}{\partial t} + v^\infty \cdot \nabla v^\infty = \frac{1}{\rho_0} \nabla p^\infty
\]
for some \( p^\infty \in L^\infty([0, T_0], H^{s+1}) \) and by the convergence in the divergence-free part of \( L^2 \) we can show the required convergence.

Notice that when \( N = 3 \) we have an analogous result to that of Theorem 2.1. We end this section with a continuation result.

**Theorem 2.15.** Under the assumptions of Theorem 2.14, the interval \([0, T_\ast)\) with \( T_\ast < \infty \) is a maximal interval of \( H^s \) existence if and only if
\[
\max_{1 \leq i, j \leq N} \left\| \frac{\partial v^\infty}{\partial x_j} \right\|_{L^\infty} \to \infty, \text{ as } t \nearrow T_\ast
\]
(86)

For a proof see [6]. This result gives a hint to the vorticity result (Theorem 4.1).

## 3 The Cauchy Problem - A Direct Approach

In this section we concern ourselves with the Cauchy problem of the 3-dimensional incompressible Euler equation in whole space. These read
\[
\begin{align*}
\partial_t u + u \cdot \nabla u &= -\nabla p \\
\text{div } u &= 0 \\
\end{align*}
\]
\[u(t, \cdot) = u_0\]
for the velocity field \( u \) and pressure field \( p \), which, for \( t \in \mathbb{R} \) sufficiently close to zero, should define functions \( u(t) : \mathbb{R}^3 \to \mathbb{R}^3 \) and \( p(t) : \mathbb{R}^3 \to \mathbb{R} \) that satisfy the Euler equations in some sense. Our presentation follows that of [7].

The goal is to prove a local existence result. To this end, a fruitful strategy is to view solutions to the Euler equation as flows on some function, and to invoke a fixed point theorem to guarantee local existence and uniqueness of such flows. We shall use the following generalisation of the standard Picard theorem of existence and uniqueness of solutions to ODE in finite-dimensional normed spaces.

**Theorem 3.1.** Let \( X \) be a Banach space and \( U \subseteq X \) be an open subset. Suppose \( F : U \to X \) is a locally Lipschitz mapping.

Then for any \( x_0 \in U \) there exists \( T > 0 \) such that the following ODE in \( X \),
\[
\frac{dx}{dt} = F(x), \quad x(0) = x_0
\]
has a unique solution \( x \in C^1([-T,T], U) \).

The proof is an obvious adaptation of the standard proof using the contraction mapping theorem, so we shall omit it here.
3.1 Removal of the Pressure Term

An awkward feature of the Euler equation (and also of the Navier-Stokes equation) is the presence of the pressure term, which is not governed by a dynamical equation. In fact, pressure is not really an independent variable, but serves a role analogous to a Lagrange multiplier enforcing the incompressibility condition $\text{div } u = 0$. Because of this, it is more natural to work in the Sobolev spaces of divergence-free vector fields,

$$V^s(\mathbb{R}^d) := \left\{ v \in H^m(\mathbb{R}^d, \mathbb{R}^d) \mid \text{div } v = 0 \right\}, \quad s \in \mathbb{R}_{\geq 0}.$$  

More precisely, $V^s(\mathbb{R}^d)$ is the closure in $H^s(\mathbb{R}^d)$ of the $C^\infty$ divergence-free vector fields.

**Lemma 3.2 (Hodge decomposition).** For $s \geq 0$, every vector field $v \in H^s(\mathbb{R}^d, \mathbb{R}^d)$ has a unique orthogonal decomposition

$$v = w + \nabla \varphi$$

with $w \in V^s(\mathbb{R}^d)$ and $\varphi \in S'(\mathbb{R}^d)$, $\nabla \varphi \in H^s(\mathbb{R}^d)$.

We therefore define the Leray projection operator $P : H^s(\mathbb{R}^d, \mathbb{R}^d) \to V^s(\mathbb{R}^d)$.

**Proof.** We first consider uniqueness. Since $u$ is divergence-free, the Fourier transform of $w$ must satisfy

$$\xi \perp F[w](\xi) \quad \forall \xi \in \mathbb{R}^d.$$  

The orthogonality condition then forces

$$F[w](\xi) = F[v](\xi) - \frac{\xi \cdot F[v](\xi)}{\|\xi\|^2} \xi.$$  

This proves uniqueness of $w$ and hence of $\varphi$.

For existence, we simply define $w$ and $\varphi$ by their Fourier transforms,

$$F[\varphi](\xi) := -i \frac{\xi \cdot F[v](\xi)}{\|\xi\|^2}, \quad F[w](\xi) := F[v](\xi) - \frac{\xi \cdot F[v](\xi)}{\|\xi\|^2} \xi.$$  

It is easy to check that this works.  

**Remark 3.3.** The Leray projection operator $P$ has the following properties:

(i) $P$ commutes with distributional derivatives: For $m \in \mathbb{Z}_{\geq 0}$,

$$P D^\alpha = D^\alpha P \text{ on } H^m(\mathbb{R}^d), \quad \text{for } |\alpha| \leq m.$$  

(ii) $P$ commutes with mollifiers $J_\varepsilon$ (see the next section for the definition),

$$P J_\varepsilon = J_\varepsilon P \text{ on } H^m(\mathbb{R}^d), \quad \text{for } \varepsilon > 0.$$  

The verification of these properties is straightforward. We shall make use of them without explicit citation.

Using Leray projection, we can eliminate the pressure term to obtain a dynamical equation for the velocity field only. We will then concentrate on solving the resulting pressureless equation. In other words, we look for solutions in the following sense.

**Definition 3.4.** Let $m \in \mathbb{Z}$ with $m \geq 3$, let $u_0 \in V^m(\mathbb{R}^3)$ and let $T > 0$. If

$$u \in C^0((-T,T), V^m(\mathbb{R}^3)) \cap C^1((-T,T), V^{m-1}(\mathbb{R}^3))$$  

satisfies the pressureless Euler equation,

$$\frac{\partial u}{\partial t} + P (u \cdot \nabla u) = 0$$  

$$u(0) = u_0,$$  

we say $u$ is a strong solution to the Euler equation in $V^m(\mathbb{R}^3)$ up to time $T$.  

Remark 3.5.

(i) Given a strong solution \( u \), we can recover the pressure by orthogonally projecting \((u \cdot \nabla)u\) to the space of gradients via Hodge decomposition (note that \( \partial_t u \) is a divergence-free vector field and so has no gradient component under Hodge decomposition).

(ii) We impose \( m \geq 3 \) in the definition in order to ensure, using Lemma 3.10 below, \((u \cdot \nabla)u \in C^0((-T, T), H^{m-1}(\mathbb{R}^3))\). \( \square \)

Moreover, we have the following uniqueness result.

**Theorem 3.6.** Let \( m \in \mathbb{Z} \) with \( m \geq 3 \), let \( T > 0 \), and let \( u_0 \in V^m(\mathbb{R}^3) \). Then there can be at most one strong solution \( u \) with initial data \( u_0 \) up to time \( T \).

**Proof.** Suppose \( u_1, u_2 \) are any two strong solutions up to time \( T \), with the same initial data \( u_0 \).
Denote \( w := u_1 - u_2 \). Then
\[
\frac{d}{dt} \left( \frac{1}{2} \|w\|_{H^0}^2 \right) = - \int_{\mathbb{R}^3} [u_1 \cdot \nabla u_1 - u_2 \cdot \nabla u_2] \cdot w \, dx
\]
\[
= - \int_{\mathbb{R}^3} [u_1 \cdot \nabla w + w \cdot \nabla u_2] \cdot w \, dx
\]
Since
\[
\int_{\mathbb{R}^3} [u_1 \cdot \nabla w] \cdot w \, dx = \sum_{i=1}^3 \int_{\mathbb{R}^3} (u_1 \cdot \nabla w_i) w_i \, dx = \sum_{i=1}^3 \int_{\mathbb{R}^3} u_1 \cdot \nabla \left( \frac{1}{2} w_i^2 \right) \, dx
\]
\[
= - \int_{\mathbb{R}^3} \frac{1}{2} |w|^2 \text{div} u_1 \, dx = 0,
\]
we have
\[
\frac{d}{dt} \left( \frac{1}{2} \|w\|_{H^0}^2 \right) = - \int_{\mathbb{R}^3} [w \cdot \nabla u_2] \cdot w \, dx \leq \int_{\mathbb{R}^3} |w|^2 |\nabla u| \, dx
\]
\[
\leq \|\nabla u\|_{L^\infty} \|w\|_{H^0}^2.
\]
Using Gronwall’s inequality, we find
\[
\|w(t)\|_{H^0} \leq \|w(0)\|_{H^0} \exp \left( 2 \int_0^t \|\nabla u(s)\|_{L^\infty} \, ds \right) = 0
\]
as required. \( \square \)

Therefore the notion of a strong solution seems to be a reasonably good one to work in.

### 3.2 Regularisation by Mollifiers

The Euler equation contains spatial derivative terms, and thus the vector fields generating the flow are unbounded on Sobolev spaces (including the Lebesgue spaces). As such we cannot directly apply Theorem 3.1 to the Euler equation. To overcome this problem, we shall initially work with a regularisation of the pressureless Euler equation.

Specifically, define the non-negative smooth function \( \eta : \mathbb{R}^d \to \mathbb{R} \) by
\[
\eta(x) := \begin{cases} 
\text{const} \cdot \exp \left( -\frac{1}{\text{const} \cdot |x|} \right), & \text{if } |x| \leq 1 \\
0, & \text{otherwise}
\end{cases}
\]
where the positive constant is chosen so that
\[
\int_{\mathbb{R}^d} \eta(x) \, dx = 1.
\]
For \( \varepsilon > 0 \) we define the smooth function \( \eta_\varepsilon : \mathbb{R}^d \to \mathbb{R} \) by
\[
\eta_\varepsilon(x) := \varepsilon^{-d} \eta \left( \frac{x}{\varepsilon} \right),
\]
and we define the mollifier \( \mathcal{J}_\varepsilon : L^1_{\text{loc}}(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d) \) to be the convolution
\[
\mathcal{J}_\varepsilon f := \eta_\varepsilon \ast f.
\]
We will use standard properties of these mollifiers without citation; these and their proofs may be found, for example, in [4], Chapter 5 and Appendix C. We also remind the reader (see Remark 3.3) that Leray projection commutes with mollifiers.

We consider the following regularisation of the pressureless Euler equation.
\[
\frac{\partial}{\partial t} u^\varepsilon + \mathcal{P} \mathcal{J}_\varepsilon [(\mathcal{J}_\varepsilon u^\varepsilon) \cdot \nabla (\mathcal{J}_\varepsilon u^\varepsilon)] = 0
\]
\[
u^\varepsilon(0, \cdot) = u_0
\]
This equation arises from applying Leray projection to the pressureless Euler equation (87). We will prove global existence and uniqueness for the Cauchy problem (88); see Corollary 3.13 below.

To end off this section, we prove the following estimates which we will need in the sequel.

**Lemma 3.7.** For \( m, k \in \mathbb{Z}_{\geq 0} \) and \( 0 < \varepsilon < 1 \), we have the estimates
\[
\| \mathcal{J}_\varepsilon f \|_{H^{m+k}} \leq \frac{C(d, m, k)}{\varepsilon^k} \| f \|_{H^m},
\]
\[
\| \mathcal{J}_\varepsilon D^k f \|_{L^\infty} \leq \frac{C(d, k)}{\varepsilon^{d/2+k}} \| f \|_{H^0}.
\]
**Proof.** For \( f \in H^m(\mathbb{R}^d) \) and \( \alpha, \beta \in \mathbb{Z}^d_{\geq 0} \) with \( |\alpha| \leq k \) and \( |\beta| \leq m \), we have
\[
D^{\alpha+\beta} \mathcal{J}_\varepsilon f = (D^\alpha \eta_\varepsilon) \ast (D^\beta f).
\]
Using the Schwarz inequality, we find
\[
\left\| D^{\alpha+\beta} \mathcal{J}_\varepsilon f \right\|^2_{L^2} = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} D^\alpha \eta_\varepsilon(y) \ D^\beta f(x-y) \ dy \right|^2 \ dx
\]
\[
\leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |D^\alpha \eta_\varepsilon(y)| |D^\beta f(x-y)| \ dy \right) \left( \int_{\mathbb{R}^d} |D^\alpha \eta_\varepsilon(y)| \ dy \right) \ dx
\]
\[
= \|D^\alpha \eta_\varepsilon\|^2_{L^1} \|D^\beta f\|^2_{L^2}
\]
so that
\[
\| D^{\alpha+\beta} \mathcal{J}_\varepsilon f \|_{L^2} \leq \|D^\alpha \eta_\varepsilon\|_{L^1} \|D^\beta f\|_{L^2} = \frac{\|D^\alpha \eta\|_{L^1}}{\varepsilon^{d/2+|\alpha|}} \|D^\beta f\|_{L^2}
\]
Summing over \( \alpha \) and \( \beta \) proves the first assertion.

The second assertion is even easier: For \( |\alpha| \leq k \), the Schwarz inequality gives immediately
\[
\| \mathcal{J}_\varepsilon D^\alpha f \|_{L^\infty} \leq \| (D^\alpha \eta_\varepsilon) \ast f \|_{L^\infty} \leq \|D^\alpha \eta_\varepsilon\|_{L^1} \| f \|_{L^2} = \frac{\|D^\alpha \eta\|_{L^2}}{\varepsilon^{d/2+|\alpha|}} \| f \|_{L^2}
\]
As before, sum over \( \alpha \) to obtain the result. \( \square \)
Lemma 3.8. For $m \in \mathbb{Z}_{\geq 1}$, there is a constant $C = C(d, m) > 0$ such that

$$\|J_\varepsilon f - f\|_{H^{m-1}} \leq C\varepsilon \|f\|_{H^m}$$

holds for all $f \in H^m(\mathbb{R}^d)$.

Proof. We first consider the case $m = 1$. For $\theta > 0$ we have

$$J_\varepsilon J_\theta f(x) - J_\theta f(x) = \int_{B(0, \varepsilon)} \eta_\varepsilon(z) (J_\varepsilon f(x - z) - J_\varepsilon f(x)) \, dz$$

$$= \int_{B(0, \varepsilon)} \eta_\varepsilon(z) \int_0^1 z \cdot \nabla f(x - sz) \, ds \, dz,$$

so that, using Hölder’s inequality, we get

$$|J_\varepsilon J_\theta f(x) - J_\theta f(x)| \leq \varepsilon \int_{B(0, \varepsilon)} \int_0^1 \eta_\varepsilon(z) |\nabla f(x - sz)| \, ds \, dz$$

$$= \varepsilon \left[ \int_{B(0, \varepsilon)} \int_0^1 \eta_\varepsilon(z) (\nabla f(x - sz))^2 \, ds \, dz \right]^{1/2}.$$ 

Squaring both sides and integrating over $\mathbb{R}^d$, we obtain

$$\|J_\varepsilon J_\theta f - J_\theta f\|_{L^2} \leq \varepsilon \|\nabla f\|_{L^2} \leq C\varepsilon \|f\|_{H^1}.$$

Since $\theta > 0$ was arbitrary, taking $\theta \to 0$ gives the assertion for $m = 1$.

The case $m > 1$ is proved by applying the above argument to each of the weak derivatives of $f$ up to order $m - 1$. $lacksquare$

3.3 Global Existence and Uniqueness for the Regularised Pressureless Equation

The goal of this subsection is to prove a global existence and uniqueness result for the regularised pressureless equation (88). Here the term “global” refers to the solution existing for all time $t \in \mathbb{R}$. Due to mollification, the vector fields in (88) are now locally Lipschitz, albeit with the Lipschitz constant depending as negative powers of the regularisation parameter $\varepsilon$. Despite this, the Picard theorem (Theorem 3.1) will still apply to give local existence and uniqueness. After this we can patch these local solutions together to form a maximal solution, and then prove that this maximal solution is necessarily a global solution.

First we need a preliminary estimate.

Lemma 3.9. For $m \in \mathbb{Z}_{\geq 0}$, there is a constant $C = C(d, m) > 0$ such that for all $u, v \in L^\infty \cap H^m(\mathbb{R}^d)$,

$$\|uv\|_{H^m} \leq C (\|u\|_{L^\infty} \|v\|_{H^m} + \|u\|_{H^m} \|v\|_{L^\infty}).$$

Proof. The claim is trivial if $m = 0$, so we focus on the case $m \geq 1$. Let $\alpha \in \mathbb{Z}_{\geq 0}^d$ with $|\alpha| \leq m$.

If $|\alpha| = 0$, then clearly

$$\|D^\alpha (uv)\|_{L^2} = \|uv\|_{L^2} \leq \|u\|_{L^\infty} \|v\|_{L^2} + \|u\|_{L^2} \|v\|_{L^\infty}.$$ 

Otherwise, for $1 \leq |\alpha| \leq m$, we have, using Leibniz’s differentiation formula and Hölder’s inequality,

$$\|D^\alpha (uv)\|_{L^2} \leq C(d, m) \sum_{\beta \leq \alpha} \|D^\beta u \, D^{\alpha - \beta} v\|_{L^2}$$

$$\leq C(d, m) \sum_{\beta \leq \alpha} \|D^\beta u\|_{L^2(\mathbb{R}^d)} \|D^{\alpha - \beta} v\|_{L^2(\mathbb{R}^d)}.$$
By the Gagliardo-Nirenberg interpolation inequality,
\[
\|D^\alpha u\|_{L^2(m/|\beta|)} \leq C(d, m) \|u\|_{L^\infty}^{1-|\beta|/|\alpha|} \|D^\beta u\|_{L^2}^{|\beta|/|\alpha|},
\]
\[
\|D^{\alpha - \beta} v\|_{L^2(m/|\alpha - \beta|)} \leq C(d, m) \|v\|_{L^\infty}^{1-|\beta|/|\alpha|} \|D^\beta v\|_{L^2}^{1-|\beta|/|\alpha|}.
\]
Plugging these into our estimates above and using Young’s inequality, we find, for 1 \leq |\alpha| \leq m,
\[
\|D^{\alpha}(uv)\|_{L^2} \leq C(d, m) \sum_{\beta \leq \alpha} \left(\|u\|_{L^\infty} \|D^\beta v\|_{L^2}\right)^{1-\frac{|\beta|}{|\alpha|}} \left(\|v\|_{L^\infty} \|D^\beta u\|_{L^2}\right)^{\frac{|\beta|}{|\alpha|}} \leq C(d, m) \left(\|u\|_{L^\infty} \|D^\alpha v\|_{L^2} + \|v\|_{L^\infty} \|D^\alpha u\|_{L^2}\right).
\]
The result now follows by summing over all \alpha \in \mathbb{Z}_{\geq 0} with 0 \leq |\alpha| \leq m. \square

From the preceding result, it is easy to see that we have a Banach algebra structure on \(H^m(\mathbb{R}^d)\) whenever \(m > d/2\). We state it as a lemma here, along with an immediate corollary regarding uniqueness of strong solutions to the Euler equation. We will make heavy use of this estimate in the next subsection.

Lemma 3.10. For \(m \in \mathbb{Z}, m > d/2\), there exists a constant \(C = C(d, m) > 0\) such that
\[
\|uv\|_{H^m} \leq C \|u\|_{H^m} \|v\|_{H^m} \quad \text{for all } u, v \in H^m(\mathbb{R}^d).
\]
In other words, \(H^m(\mathbb{R}^d)\) is a Banach algebra.

Proof. This is immediate from the previous lemma and the Sobolev inequality \(\|\cdot\|_{L^\infty} \leq C(d, m)\|\cdot\|_{H^m} \) for \(H^m(\mathbb{R}^d)\) when \(m > d/2\). \square

Theorem 3.11. Let \(m \in \mathbb{Z}_{\geq 0}\) and \(u_0 \in V^m(\mathbb{R}^3)\), and let \(\varepsilon > 0\). Then
(i) There exists a unique solution \(u^\varepsilon \in C^1([-T, T], V^m(\mathbb{R}^3))\) to (88), for some \(T = T(||u_0||_{H^m}, \varepsilon)\).
(ii) For \(T > 0\) and \(u^\varepsilon \in C^1([-T, T], V^0(\mathbb{R}^3))\) a solution to (88), we have
\[
\|u^\varepsilon(t)\|_{H^m} = ||u_0||_{H^m} \quad \text{for all } t \in [-T, T].
\]

Proof. We write (88) in the form \(\partial_t u^\varepsilon = F_\varepsilon(u^\varepsilon)\), where \(F_\varepsilon : V^m(\mathbb{R}^3) \rightarrow V^m(\mathbb{R}^3)\) is given by
\[
F_\varepsilon(v) := -\mathcal{P} \mathcal{J}_\varepsilon (\langle \mathcal{J}_\varepsilon v, \nabla (\mathcal{J}_\varepsilon v) \rangle).
\]
For \(v_1, v_2 \in V^m(\mathbb{R}^3)\), we have, from Lemma 3.9, the estimate
\[
\|F_\varepsilon(v_1) - F_\varepsilon(v_2)\|_{H^m} \leq \|\mathcal{P} \mathcal{J}_\varepsilon \|_{L^\infty} \|\mathcal{J}_\varepsilon v_1 \cdot \nabla \mathcal{J}_\varepsilon (v_1 - v_2)\|_{H^m} + ||\mathcal{P} \mathcal{J}_\varepsilon \|_{L^\infty} \|\mathcal{J}_\varepsilon (v_1 - v_2) \cdot \nabla \mathcal{J}_\varepsilon v_2\|_{H^m}
\]
\[
\leq C \|\mathcal{J}_\varepsilon v_1\|_{L^\infty} \|\mathcal{J}_\varepsilon \nabla (v_1 - v_2)\|_{H^m} + ||\mathcal{J}_\varepsilon v_1\|_{H^m} \|\mathcal{J}_\varepsilon \nabla (v_1 - v_2)\|_{L^\infty} + \|\mathcal{J}_\varepsilon (v_1 - v_2)\|_{L^\infty} \|\mathcal{J}_\varepsilon \nabla v_2\|_{H^m} + \|\mathcal{J}_\varepsilon (v_1 - v_2)\|_{H^m} \|\mathcal{J}_\varepsilon \nabla v_2\|_{L^\infty}
\]
By Lemma 3.7 we have the bound
\[
\|F_\varepsilon(v_1) - F_\varepsilon(v_2)\|_{H^m} \leq \frac{C}{\varepsilon^{d/2 + m}} (\|v_1\|_{H^m} + \|v_2\|_{H^m}) \|v_1 - v_2\|_{H^m}
\]
(89)
where \(C = C(d, m) > 0\).

Thus \(F_\varepsilon\) is locally Lipschitz, so Theorem 3.1 guarantees the existence and uniqueness of a flow \(u^\varepsilon \in C^1([-T, T], V^m)\), for some \(T = T(||u_0||_{H^m}, \varepsilon) > 0\). The proof of (i) is complete.
Next, take the $H^0$ inner product of (88) with the solution $u^\varepsilon$ to obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \| u^\varepsilon \|_{H^0}^2 \right) = - \int_{\mathbb{R}^3} u^\varepsilon \cdot \mathcal{P} \mathcal{J}_\varepsilon \left[ (\mathcal{J}_\varepsilon u^\varepsilon) \cdot \nabla (\mathcal{J}_\varepsilon u^\varepsilon) \right] dx
\]
\[
= - \int_{\mathbb{R}^3} (\mathcal{J}_\varepsilon u^\varepsilon) \cdot \left[ (\mathcal{J}_\varepsilon u^\varepsilon) \cdot \nabla (\mathcal{J}_\varepsilon u^\varepsilon) \right] dx
\]
\[
= - \frac{1}{2} \int_{\mathbb{R}^3} (\mathcal{J}_\varepsilon u^\varepsilon) \cdot \nabla \left( (\mathcal{J}_\varepsilon u^\varepsilon)^2 \right) dx
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^3} \left( \text{div} (\mathcal{J}_\varepsilon u^\varepsilon) \right) (\mathcal{J}_\varepsilon u^\varepsilon)^2 dx
\]
\[= 0. \]
This proves (ii). $\square$

As with ODE in normed spaces, the Picard theorem (Theorem 3.1) allows us to patch local-in-time solutions to give, for each initial data $u_0 \in H^m(\mathbb{R}^3)$, a unique maximal solution $u^\varepsilon \in C^1((T_-(u_0), T_+(u_0)), V^m(\mathbb{R}^3))$ to the regularised pressureless equation (88). We now show that, in fact, the maximal solution is global, i.e. $(T_-(u_0), T_+(u_0)) = (-\infty, \infty)$.

For this, we need the following general result for ODE on Banach spaces.

**Theorem 3.12.** Let $X$ be a Banach space and $U \subseteq X$ be an open subset, and let $F : U \rightarrow X$ be locally Lipschitz. Suppose $x \in C^1((a, b), X)$ is the maximal solution to the ODE
\[
\frac{dx}{dt} = F(x), \quad x(0) = x_0 \in U.
\]
If $b < \infty$, then at least one of the following holds.

(i) $\limsup_{t \uparrow b} \| F(x(t)) \|_X = +\infty$.

(ii) $\lim_{t \uparrow b} x(t)$ exists and does not belong to $U$.

The obvious analogous statement apply in the case $a > -\infty$.

**Proof.** Assume $b < \infty$ and (i) does not hold; we have to prove (ii). Since (i) does not hold, we have
\[
M := 1 + \limsup_{t \uparrow b} \| F(x(t)) \|_X < \infty
\]
so that there exists some sufficiently small $\delta > 0$ such that
\[
\| x(t) - x(s) \|_X \leq \int_s^t \| F(x(\tau)) \|_X d\tau
\]
\[\leq M |t - s| \]
whenever $b - \delta < s \leq t < b$.

This shows the existence of $x(b-):= \lim_{t \uparrow b} x(t)$. If $x(b-) \in U$, then we could apply Theorem 3.1 to find a local-in-time solution starting from $x(b-)$; patching this solution to $x$ gives a solution which exists beyond time $b$, which contradicts the maximality of $x$. $\square$

**Corollary 3.13.** The unique maximal solution $u \in C^1((a, b), V^m(\mathbb{R}^3))$ to the regularised pressureless equation (88) is global, i.e. $a = -\infty$ and $b = +\infty$.

**Proof.** Using Equation (89) with $v_1 = u^\varepsilon(t), v_2 = 0$, we find
\[
\frac{d}{dt} \left( \| u^\varepsilon(t) \|_{H^m}^2 \right) \leq 2 \| u^\varepsilon(t) \|_{H^m} \| F_\varepsilon(u^\varepsilon(t)) \|_{H^m}
\]
\[\leq \frac{C(d,m)}{\varepsilon^{5/2+\alpha}} \| u_0 \|_{H^0} \| u^\varepsilon(t) \|_{H^m}^2. \]

By Gronwall’s inequality,
\[ \|u^\varepsilon(t)\|^2_{H^m} \leq \|u_0\|^2_{H^m} \exp\left( \frac{C(d,m)}{\varepsilon^{5/2+m}} \|u_0\|_{H^0} |t| \right) \]
for all \( t \geq 0 \).

By solving (88) we obtain a similar estimate for negative times. Hence, in fact,
\[ \|u^\varepsilon(t)\|^2_{H^m} \leq \|u_0\|^2_{H^m} \exp\left( \frac{C(d,m)}{\varepsilon^{5/2+m}} \|u_0\|_{H^0} |t| \right) \]
for all \( t \in \mathbb{R} \).

Now we apply Theorem 3.12 with \( X = V^m(\mathbb{R}^3) \) and \( U = X \); since neither of the conclusions hold for any finite \( b > 0 \), we must have \( b = +\infty \). A similar argument gives \( a = -\infty \). \( \square \)

### 3.4 Local Existence to the Euler Equation

Thus far we have proved global existence and uniqueness to the regularised pressureless Euler equation (88) for each \( \varepsilon > 0 \). Much of this subsection will be devoted to showing that we can take the limit \( \varepsilon \to 0 \).

**Lemma 3.14.** For \( m \in \mathbb{Z}_{\geq 1} \) there is a constant \( C = C(d,m) > 0 \) such that for all \( u, v \in L^\infty \cap H^m(\mathbb{R}^d) \),
\[ \sum_{|\alpha| \leq m} \| D^\alpha (uv) - uD^\alpha v \|_{L^2} \leq C \left( \| \nabla u \|_{L^\infty} \| v \|_{H^{m-1}} + \| u \|_{H^m} \| v \|_{L^\infty} \right) . \]

**Proof.** Observe that
\[ \sum_{|\alpha| \leq m} \| D^\alpha (uv) - uD^\alpha v \|_{L^2} \leq C(d,m) \sum_{|\beta| \leq m-1} \| D^\beta (\nabla u) v \|_{L^2} \]
\[ \leq C(d,m) \| (\nabla u) v \|_{H^{m-1}} . \]

By Lemma 3.9,
\[ \| (\nabla u) v \|_{H^{m-1}} \leq C(d,m) (\| \nabla u \|_{L^\infty} \| v \|_{H^{m-1}} + \| \nabla u \|_{H^{m-1}} \| v \|_{L^\infty}) \]
\[ \leq C(d,m) (\| \nabla u \|_{L^\infty} \| v \|_{H^{m-1}} + \| u \|_{H^m} \| v \|_{L^\infty}) . \]

This completes the proof. \( \square \)

**Lemma 3.15.** Let \( m \in \mathbb{Z}_{\geq 1} \) and \( u_0 \in V^m(\mathbb{R}^3) \). Then the unique solution \( u^\varepsilon \in C^1(\mathbb{R}, V^m(\mathbb{R}^3)) \) to (88) satisfies the estimate
\[ \frac{1}{2} \frac{d}{dt} \left( \| u^\varepsilon \|^2_{H^m} \right) \leq C \| \mathcal{J} \nabla u^\varepsilon \|_{L^\infty} \| u^\varepsilon \|^2_{H^m} \]
for some constant \( C = C(m) > 0 \).

**Proof.** For \( \alpha \in \mathbb{Z}_{\geq 0}^3, |\alpha| \leq m \), we take the \( \alpha \)-th derivative of (88), and then take the inner product with \( D^\alpha u^\varepsilon \). This gives
\[ \frac{1}{2} \frac{d}{dt} \left( \| D^\alpha u^\varepsilon \|^2_{L^2} \right) = - \langle D^\alpha u^\varepsilon, D^\alpha \mathcal{J} \left[ ([\mathcal{J} u^\varepsilon] \cdot \nabla (\mathcal{J} u^\varepsilon)) \right] \rangle_{L^2} \]
\[ = - \langle D^\alpha \mathcal{J} u^\varepsilon, D^\alpha \left[ [\mathcal{J} u^\varepsilon] \cdot \nabla (\mathcal{J} u^\varepsilon) \right] \rangle_{L^2} \]
\[ = - \langle D^\alpha \mathcal{J} u^\varepsilon, D^\alpha \left[ [\mathcal{J} u^\varepsilon] \cdot \nabla (\mathcal{J} u^\varepsilon) \right] - [\mathcal{J} u^\varepsilon] \cdot \nabla (D^\alpha \mathcal{J} u^\varepsilon) \rangle_{L^2} \]
\[ = - \langle D^\alpha \mathcal{J} u^\varepsilon, (\mathcal{J} u^\varepsilon) \cdot \nabla (D^\alpha \mathcal{J} u^\varepsilon) \rangle_{L^2} \]

The last term on the right vanishes due to the divergence theorem:
\[ \langle D^\alpha \mathcal{J} u^\varepsilon, (\mathcal{J} u^\varepsilon) \cdot \nabla (D^\alpha \mathcal{J} u^\varepsilon) \rangle_{L^2} = \int_{\mathbb{R}^3} \frac{1}{2} \langle \mathcal{J} u^\varepsilon \cdot \nabla (\mathcal{J} u^\varepsilon) \rangle_{L^2} dx \]
\[ = - \int_{\mathbb{R}^3} \frac{1}{2} |\text{div} (\mathcal{J} u^\varepsilon) | \| \mathcal{J} u^\varepsilon \|^2_{L^2} dx = 0 . \]
Now, summing our estimate over $\alpha$ and applying Lemma 3.14 gives
\[
\frac{1}{2} \frac{d}{dt} \left( \|u^\varepsilon\|^2_{H^m} \right) \\
\leq \|u^\varepsilon\|_{H^m} \sum_{|\alpha| \leq m} \|D^\alpha [(\mathcal{J}_\varepsilon u^\varepsilon) \cdot \nabla (\mathcal{J}_\varepsilon u^\varepsilon)] - (\mathcal{J}_\varepsilon u^\varepsilon) \cdot \nabla (D^\alpha \mathcal{J}_\varepsilon u^\varepsilon)\|_{L^2} \\
\leq C(m) \|u^\varepsilon\|_{H^m} \left( \|\nabla \mathcal{J}_\varepsilon u^\varepsilon\|_{L^\infty} \|\nabla \mathcal{J}_\varepsilon u^\varepsilon\|_{H^{m-1}} + \|\mathcal{J}_\varepsilon u^\varepsilon\|_{H^m} \|\nabla \mathcal{J}_\varepsilon u^\varepsilon\|_{L^\infty} \right) \\
\leq C(m) \|\mathcal{J}_\varepsilon \nabla u^\varepsilon\|_{L^\infty} \|u^\varepsilon\|^2_{H^m}
\]
as required.

**Proposition 3.16.** Let $m \in \mathbb{Z}$ with $m \geq 3$. Then the solution $u^\varepsilon$ to (88) with initial data $u_0 \in V^m(\mathbb{R}^3)$ satisfies
\[
\|u^\varepsilon(t)\|_{H^m} \leq \frac{\|u_0\|_{H^m}}{1 - C_1 |t| \|u_0\|_{H^m}} \quad \text{for } |t| < \frac{1}{C_1 \|u_0\|_{H^m}}
\]
where $C_1 = C_1(m) > 0$ is any constant satisfying Lemma 3.15.

**Proof.** We first consider the case $t \geq 0$. Since $m \geq 3$, Sobolev embedding gives
\[
\|\mathcal{J}_\varepsilon \nabla u^\varepsilon\|_{L^\infty} \leq \|\nabla u^\varepsilon\|_{L^\infty} \leq \|\nabla u^\varepsilon\|_{H^{m-1}} \leq \|u^\varepsilon\|_{H^m}.
\]
Thus Lemma 3.15 gives, for any $\theta > 0$,
\[
\frac{1}{2} \frac{d}{dt} \left( \|u^\varepsilon\|^2_{H^m} \right) \leq C_1 \|u^\varepsilon\|^3_{H^m} \leq C_1 \left( \theta + \|u^\varepsilon\|^2_{H^m} \right) \frac{3}{2}.
\]
Therefore, for $0 \leq t < 1/(C_1 \sqrt{\theta + \|u_0\|^2_{H^m}})$, we have, by direct integration,
\[
\frac{1}{\sqrt{\theta + \|u_0\|^2_{H^m}}} - \frac{1}{\sqrt{\theta + \|u^\varepsilon(t)\|^2_{H^m}}} \leq C_1 t,
\]
so that
\[
\sqrt{\theta + \|u^\varepsilon(t)\|^2_{H^m}} \leq \frac{\sqrt{\theta + \|u^\varepsilon(t)\|^2_{H^m}}}{1 - C_1 t \sqrt{\theta + \|u^\varepsilon(t)\|^2_{H^m}}}.
\]
Since $\theta > 0$ was arbitrary, the limit $\theta \to 0$ gives the assertion for $t \geq 0$. The case $t < 0$ is proved similarly by applying the above argument to the backwards solution of (88). \qed

**Proposition 3.17.** Fix initial data $u_0 \in V^3(\mathbb{R}^3)$. Then the family $\{u^\varepsilon\}_0<\varepsilon<1$ of approximate solutions to (88), with the same initial data $u_0$, satisfies
\[
\|u^\varepsilon(t) - u^0(t)\|_{H^0} \leq C_1 C_2 (\varepsilon + \theta) \frac{\|u_0\|^2_{H^3} |t|}{(1 - C_1 |t| \|u_0\|_{H^3})^{C_2}}
\]
whenever
\[
|t| < T_1 := \frac{1}{C_1 \|u_0\|_{H^3}},
\]
where $C_1 > 0$ is any constant satisfying Lemma 3.15, and $C_2 > 2$ is some constant depending on our choice of $C_1$, but not on $\varepsilon, \theta$ or $t$.

**Proof.** First suppose $t \geq 0$. From (88) it is immediate that
\[
\frac{d}{dt} \left( \|u^\varepsilon - u^0\|^2_{H^m} \right) \\
= 2 \left( \mathcal{J}_\varepsilon [(\mathcal{J}_\varepsilon u^\varepsilon) \cdot \nabla (\mathcal{J}_\varepsilon u^\varepsilon)] - \mathcal{J}_\theta [(\mathcal{J}_\theta u^\theta) \cdot \nabla (\mathcal{J}_\theta u^\theta)] , u^\varepsilon - u^\theta \right)_{L^2}.
\]
Now,
\[ \mathcal{J}_\varepsilon \left( (\mathcal{J}_\varepsilon u^\varepsilon) \cdot \nabla (\mathcal{J}_\varepsilon u^\varepsilon) \right) - \mathcal{J}_0 \left( (\mathcal{J}_0 u^0) \cdot \nabla (\mathcal{J}_0 u^0) \right) \]
\[ = \left( \mathcal{J}_\varepsilon - \mathcal{J}_0 \right) \left( (\mathcal{J}_\varepsilon u^\varepsilon) \cdot \nabla (\mathcal{J}_\varepsilon u^\varepsilon) \right) + \mathcal{J}_0 \left[ (\mathcal{J}_\varepsilon - \mathcal{J}_0) u^\varepsilon \right] \cdot \nabla (\mathcal{J}_\varepsilon u^\varepsilon) + \mathcal{J}_0 \left[ (\mathcal{J}_0 u^0) \cdot \nabla (\mathcal{J}_\varepsilon - \mathcal{J}_0) u^\varepsilon \right] \]
\[ + \mathcal{J}_0 \left[ (\mathcal{J}_0 u^0) \cdot \nabla (\mathcal{J}_\varepsilon u^\varepsilon) \right) \]
\[ =: A_1(t) + A_2(t) + A_3(t) + A_4(t) + A_5(t) . \]

Using Lemma 3.8, Lemma 3.9, Sobolev embedding and Proposition 3.16 in this order, we find
\[ \| A_1(t) \|_{L^2} \leq C(\varepsilon + \theta) \| (\mathcal{J}_\varepsilon u^\varepsilon) \cdot \nabla (\mathcal{J}_\varepsilon u^\varepsilon) \|_{H^1} \]
\[ \leq C(\varepsilon + \theta) \left( \| u^\varepsilon \|_{L^\infty} \| \nabla u^\varepsilon \|_{H^1} + \| u^\varepsilon \|_{H^1} \| \nabla u^\varepsilon \|_{L^\infty} \right) \]
\[ \leq C(\varepsilon + \theta) \left( \frac{\| u_0 \|_{H^3}}{1 - C_1 t \| u_0 \|_{H^3}} \right)^2 . \]

Similarly we can prove the following estimates for \( A_2 \) and \( A_4 \),
\[ \| A_2(t) \|_{L^2} \leq C(\varepsilon + \theta) \left( \frac{\| u_0 \|_{H^3}}{1 - C_1 t \| u_0 \|_{H^3}} \right)^2 , \]
\[ \| A_4(t) \|_{L^2} \leq C(\varepsilon + \theta) \left( \frac{\| u_0 \|_{H^3}}{1 - C_1 t \| u_0 \|_{H^3}} \right)^2 . \]

For \( A_3 \), we have, by Sobolev embedding and Proposition 3.16,
\[ \| A_3(t) \|_{L^2} \leq C \| \nabla \mathcal{J}_\varepsilon u^\varepsilon \|_{L^\infty} \| u^\varepsilon - u^0 \|_{L^2} \]
\[ \leq C \frac{\| u_0 \|_{H^3}}{1 - C_1 t \| u_0 \|_{H^3}} \| u^\varepsilon - u^0 \|_{L^2} . \]

Finally,
\[ (A_5(t), u^\varepsilon - u^0)_{L^2} = \frac{1}{2} \int_{\mathbb{R}^3} (\mathcal{J}_0 u^0) \cdot \nabla \left[ (\mathcal{J}_0 u^0)^2 \right] \, dx \]
\[ = \frac{1}{2} \int_{\mathbb{R}^3} \left( \text{div} (\mathcal{J}_0 u^0) \right) (\mathcal{J}_0 u^0)^2 \, dx = 0 . \]

Putting the above together, there is a constant \( C_2 > 2 \), independent of \( \varepsilon, \theta \) or \( t \), such that
\[ \frac{d}{dt} \left( \| u^\varepsilon(t) - u^0(t) \|_{H^0}^2 \right) \leq C_1 C_2 (\varepsilon + \theta) \left( \frac{\| u_0 \|_{H^3}}{1 - C_1 t \| u_0 \|_{H^3}} \right)^2 \| u^\varepsilon(t) - u^0(t) \|_{H^0} \]
\[ + C_1 C_2 \frac{\| u_0 \|_{H^3}}{1 - C_1 t \| u_0 \|_{H^3}} \| u^\varepsilon(t) - u^0(t) \|_{H^0}^2 . \]

Thus, for any \( \kappa > 0 \), we have
\[ \frac{d}{dt} \left( \left( \| u^\varepsilon(t) - u^0(t) \|_{H^0}^2 + \kappa \right)^{\frac{1}{2}} \left( 1 - C_1 t \| u_0 \|_{H^3} \right)^{C_2} \right) \leq C_1 C_2 (\varepsilon + \theta) \| u_0 \|_{H^3}^2 \left( 1 - C_1 t \| u_0 \|_{H^3} \right)^{C_2 - 2} \leq C_1 C_2 (\varepsilon + \theta) \| u_0 \|_{H^3}^2 . \]

Directly integrating, we find
\[ \left( \| u^\varepsilon(t) - u^0(t) \|_{H^0}^2 + \kappa \right)^{\frac{1}{2}} \left( 1 - C_1 t \| u_0 \|_{H^3} \right)^{C_2} \leq \sqrt{\kappa} + C_1 C_2 (\varepsilon + \theta) \| u_0 \|_{H^3}^2 t . \]

The desired result for \( t \geq 0 \) is obtained by taking the limit \( \kappa \to 0 \). For the case \( t < 0 \), apply the above argument to the backwards solution of (88). \( \square \)
In fact, the bound in Proposition 3.17 will provide bounds in higher Sobolev spaces as well, thanks to the following interpolation inequality for Sobolev spaces. With these bounds we will obtain a local existence result for strong solutions to the Euler equation.

**Proposition 3.18.** Given $s > 0$, there exists a constant $C = C(d, s) > 0$ such that

$$\|f\|_{H^r} \leq C \|f\|_{H^0}^{1-r/s} \|f\|_{H^s}^{r/s}$$

for all $f \in H^s(\mathbb{R}^d)$ whenever $0 < r < s$. $\square$

This inequality is a special case of [1], Theorem 5.2. We refer the reader there for details of the proof.

**Theorem 3.19.** Let $m \in \mathbb{Z}$ with $m \geq 4$, and suppose $u_0 \in V^m(\mathbb{R}^3)$. Let $T_1$ be as in Proposition 3.17.

Then there exists a strong solution $u$ in $V^{m-1}(\mathbb{R}^3)$ to the Euler equation up to time $T_1$, such that $u^\varepsilon(t) \to u(t)$ in $V^{m-1}(\mathbb{R}^3)$, locally uniformly in time $t$, where $u^\varepsilon$ is the unique global solution to the regularised equation (88).

**Proof.** Let $T \in (0, T_1)$. By Propositions 3.18 and 3.16 respectively, we have, for any $t \in [-T,T],$

$$\|u^\varepsilon(t) - u^\theta(t)\|_{H^{m-1}} \leq C(m) \|u^\varepsilon(t) - u^\theta(t)\|_{H^0}^{1-m} \|u^\varepsilon(t) - u^\theta(t)\|_{H^m}^{m}$$

Using Proposition 3.17,

$$\sup_{t \in [-T,T]} \|u^\varepsilon(t) - u^\theta(t)\|_{H^{m-1}} \leq C(m, \|u_0\|_{H^m}, T) (\varepsilon + \theta)^{\frac{1}{m}}.$$

Therefore, for any sequence $\varepsilon_k \downarrow 0$ of regularisation parameters, $\{u^\varepsilon_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence in the Banach space $C^0([-T,T],V^{m-1}(\mathbb{R}^3))$, and thus has a limit $u \in C^0([-T,T],V^{m-1}(\mathbb{R}^3))$. It is easy to see that this limit $u$ does not depend on the choice of the sequence $\{\varepsilon_k\}_{k \in \mathbb{N}}$ of regularisation parameters, i.e. $u^\varepsilon \to u$ in $C^0([-T,T],V^{m-1}(\mathbb{R}^3))$ as $\varepsilon \to 0$.

Since $T \in (0, T_1)$ was arbitrary, we see that actually $u \in C^0((-T_1,T_1),V^{m-1}(\mathbb{R}^3))$ and $u^\varepsilon \to u$ in the topology of local uniform convergence of $C^0((-T_1,T_1),V^{m-1}(\mathbb{R}^3))$.

Next, using Lemma 3.10,

$$\|u^\varepsilon \cdot \nabla u^\varepsilon - u \cdot \nabla u\|_{H^{m-2}} \leq \|(u^\varepsilon - u) \cdot \nabla u^\varepsilon\|_{H^{m-2}} + \|u \cdot \nabla (u^\varepsilon - u)\|_{H^{m-2}}$$

so that Proposition 3.16 gives

$$\|u^\varepsilon \cdot \nabla u^\varepsilon - u \cdot \nabla u\|_{H^{m-2}} \leq \frac{C(m, \|u_0\|_{H^m})}{T_1 - |t|} \|u^\varepsilon - u\|_{H^{m-1}}.$$

Similarly, using Lemma 3.10 (this is where we need $m \geq 4$) and Lemmas 3.7 and 3.8, we have

$$\|\mathcal{J}_\varepsilon\varepsilon \cdot \nabla (\mathcal{J}_\varepsilon u^\varepsilon) - u^\varepsilon \cdot \nabla u^\varepsilon\|_{H^{m-2}}$$

so that Proposition 3.16 gives

$$\|\mathcal{J}_\varepsilon\varepsilon \cdot \nabla (\mathcal{J}_\varepsilon u^\varepsilon) - u^\varepsilon \cdot \nabla u^\varepsilon\|_{H^{m-2}} \leq \frac{C(m, \|u_0\|_{H^m})}{(T_1 - |t|)^2} \varepsilon.$$


Finally, bearing in mind Lemma 3.8 and using Lemma 3.10 and Proposition 3.16, we find

\[ \| J_\varepsilon [ (J_\varepsilon u^\varepsilon) \cdot \nabla (J_\varepsilon u^\varepsilon) ] - (J_\varepsilon u^\varepsilon) \cdot \nabla (J_\varepsilon u^\varepsilon) \|_{H^{-m-2}} \]

\[ \leq C \varepsilon \| (J_\varepsilon u^\varepsilon) \cdot \nabla (J_\varepsilon u^\varepsilon) \|_{H^{-m-1}} \]

\[ \leq C \varepsilon \| (J_\varepsilon u^\varepsilon) \|_{H^{-m}} \| \nabla (J_\varepsilon u^\varepsilon) \|_{H^{-m-1}} \]

\[ \leq C (m, \| u_0 \|_{H^m}) \varepsilon . \]

Therefore we have shown, with \( C = C(m, \| u_0 \|_{H^m}) > 0, \)

\[ \| J_\varepsilon [ (J_\varepsilon u^\varepsilon) \cdot \nabla (J_\varepsilon u^\varepsilon) ] - u \cdot \nabla u \|_{H^{-m-2}} \leq C \left( \frac{\varepsilon}{(T_1 - |t|)^2} + \frac{\| u^\varepsilon - u \|_{H^{-m-1}}}{T_1 - |t|} \right) \]

By the same argument as above, this shows that \( (J_\varepsilon u^\varepsilon) \cdot \nabla (J_\varepsilon u^\varepsilon) \to u \cdot \nabla u \) in the topology of local uniform convergence of \( C^0((-T_1, T_1), V^{m-2}(\mathbb{R}^3)) \).

Since \( u^\varepsilon \) solves the regularised pressureless Euler equation (88), we have

\[ u^\varepsilon(t) - u_0 = - \int_0^t \mathcal{P} J_\varepsilon [ (J_\varepsilon u^\varepsilon(s)) \cdot \nabla (J_\varepsilon u^\varepsilon(s)) ] \, ds , \quad \text{for } t \in (-T_1, T_1) \]

as a Bochner integral (see [10], Section V.5 for details) in \( V^{m-2}(\mathbb{R}^3) \). In the limit \( \varepsilon \to 0, \) local uniform convergence gives

\[ u(t) - u_0 = - \int_0^t \mathcal{P} [ u(s) \cdot \nabla u(s) ] \, ds , \quad \text{for } t \in (-T_1, T_1) \]

in \( V^{m-2}(\mathbb{R}^3) \). This shows that \( u \in C^1((-T_1, T_1), V^{m-2}(\mathbb{R}^3)) \), and

\[ \partial_t u = -\mathcal{P} [ u \cdot \nabla u ] , \quad u(0) = u_0 , \quad \text{in } V^{m-2}(\mathbb{R}^3) . \]

Hence \( u \) is a strong solution to the Euler equation.

\[ \square \]

Although the previous result was by no means trivial, there is one aspect in which it is rather unsatisfactory: we have required the initial data to be in \( V^m(\mathbb{R}^3) \), to prove existence of a weak solution in \( V^{m-1}(\mathbb{R}^3) \). It would have been desirable to prove an existence result such that \( u(t) \in V^m(\mathbb{R}^3) \).

The loss of spatial derivatives has its roots back in our definition of a strong solution: we have required that a strong solution should be continuous in time. It turns out that, for initial data in \( V^m(\mathbb{R}^3) \), the solution constructed in the preceding theorem does indeed describe a possibly discontinuous path in \( V^m(\mathbb{R}^3) \). We have the following result.

**Theorem 3.20.** Let \( m \in \mathbb{Z} \) with \( m \geq 4 \), and suppose \( u_0 \in V^m(\mathbb{R}^3) \). Let \( u \) be the strong solution in \( V^{m-1}(\mathbb{R}^3) \) to the Euler equation up to time \( T_1 > 0 \) constructed in Theorem 3.19. Then

\[ u \in L^\infty_{loc}((-T_1, T_1), H^m(\mathbb{R}^3)) \cap \text{Lip}_{loc}((-T_1, T_1), H^{m-1}(\mathbb{R}^3)) . \]

Moreover,

\[ C^0((-T_1, T_1), H^m_w(\mathbb{R}^3)) . \]

**Proof.** Fix \( T \in (0, T_1) \) and let \( t \in [-T, T] \). Since \( \{ u^\varepsilon(t) \}_{0 < \varepsilon < 1} \) is bounded in \( H^m(\mathbb{R}^3) \) (Proposition 3.16), for any sequence \( \{ \varepsilon_k \}_{k \in \mathbb{N}} \) with \( \varepsilon_k \to 0 \) there is a subsequence \( \{ \varepsilon_{k} \}_{k \in \mathbb{N}} \) (possibly depending on \( t \)) with \( u^{\varepsilon_k}(t) \to v_t \) for some \( v_t \in H^m(\mathbb{R}^3) \). Observe that for any \( \varphi \in C^\infty_c(\mathbb{R}^3) \) and \( \alpha \in \mathbb{Z}_{\geq 0} \) with \( |\alpha| \leq m \), we have

\[ \int_{\mathbb{R}^3} u(t) \cdot D^\alpha \varphi \, dx = \int_{\mathbb{R}^3} v_t \cdot D^\alpha \varphi \, dx . \]
This shows that $v_t = u(t)$, so $u(t) \in H^m(\mathbb{R}^3)$ and $u^\varepsilon(t) \to u(t)$. Since both $t$ and the sequence $\{\varepsilon_k\}_{k \in \mathbb{N}}$ are arbitrary, we have $u^\varepsilon(t) \to u(t)$ in $H^m(\mathbb{R}^3)$ as $\varepsilon \to 0$, for each $t \in [-T, T]$. By Proposition 3.16,

$$
\|u(t)\|_{H^m} \leq \liminf_{\varepsilon \to 0} \|u^\varepsilon(t)\|_{H^m} \leq \frac{\|u_0\|_{H^m}}{1 - C_1 T \|u_0\|_{H^m}} .
$$

Hence $u \in L^\infty([-T, T], H^m(\mathbb{R}^3))$.

Next from $\partial_t u = -P(u \cdot \nabla u)$, we have, by Lemma 3.10 and Proposition 3.16,

$$
\left\| \frac{\partial u}{\partial t} \right\|_{H^{m-1}} \leq C \|u\|_{H^{m-1}} \|\nabla u\|_{H^{m-1}} \leq C \|u\|^2_{H^m} \leq C (\|u_0\|_{H^m}, T) .
$$

Integrating, we find, for $0 \leq s < t \leq T$, that

$$
\|u(t) - u(s)\|_{H^{m-1}} \leq \int_s^t \left\| \frac{\partial u}{\partial t} (\tau) \right\|_{H^{m-1}} d\tau \leq C (\|u_0\|_{H^m}, T)(t-s) .
$$

This shows that $u \in \text{Lip}([-T, T], H^{m-1}(\mathbb{R}^3))$.

Finally, note that for any $\psi \in H^{-(m-1)}(\mathbb{R}^3)$, we have

$$
|\langle \psi, u(t) - u(s) \rangle| \leq \|\psi\|_{H^{-(m-1)}} \|u(t) - u(s)\|_{H^{m-1}} \leq C (\|u_0\|_{H^m}, T) \|\psi\|_{H^{-(m-1)}} |t-s| .
$$

Therefore $\langle \psi, u(s) \rangle \to \langle \psi, u(t) \rangle$ as $s \to t$. Since $H^{-(m-1)}(\mathbb{R}^3)$ is dense in $H^{-m}(\mathbb{R}^3)$, and $u \in L^\infty([-T, T], H^m(\mathbb{R}^3))$, we deduce that $\langle \psi, u(s) \rangle \to \langle \psi, u(t) \rangle$ as $s \to t$, for each $\psi \in H^m(\mathbb{R}^3)$, i.e. $u(s) \to u(t)$ as $s \to t$. Hence we conclude that $u \in C^0([-T, T], H^m(\mathbb{R}^3))$.

We finally arrive at the main result of this section.

**Theorem 3.21.** Let $m \in \mathbb{Z}$ with $m \geq 4$, and suppose $u_0 \in V^m(\mathbb{R}^3)$. Let $u$ be the strong solution in $V^{m-1}(\mathbb{R}^3)$ to the Euler equation up to time $T_1 > 0$ constructed in Theorem 3.19. Then

$$
u \in C^0((-T_1, T_1), V^m(\mathbb{R}^3)) \cap C^1((-T_1, T_1), V^{m-1}(\mathbb{R}^3)) .
$$

In other words, $u$ is a strong solution up to time $T_1$.

**Proof.** Since $V^m(\mathbb{R}^3)$ is a Hilbert space, it is uniformly convex. In Theorem 3.20 we have shown that $u \in C^0((-T_1, T_1), H^m_w(\mathbb{R}^3))$. If we could show that $t \mapsto \|u(t)\|_{H^m}$ is upper-semicontinuous, i.e. 

$$
\limsup_{s \to 0} \|u(t+s)\|_{H^m} \leq \|u(t)\|_{H^m} \text{ for } t \in (-T_1, T_1) ,
$$

then from uniform convexity (see [3], Proposition 3.32) it will follow that $u$ is continuous as a map $u : (-T_1, T_1) \to V^m(\mathbb{R}^3)$.

For this, fix $t \in (-T_1, T_1)$, and let $v^\varepsilon$ solve the regularised pressureless Euler equation with initial data $v^\varepsilon(0) = u(t)$. As before, we have $v^\varepsilon(t) \to v(t)$ in $H^m(\mathbb{R}^3)$, with $v \in L^\infty((-\delta, \delta), V^m(\mathbb{R}^3))$ being a strong solution of the Euler equation in $V^{m-1}(\mathbb{R}^3)$, up to some time $\delta > 0$. Local uniqueness of the strong solution (Theorem 3.6) guarantees that $v(s) = u(t+s)$. By Proposition 3.16,

$$
\|u(t+s)\|_{H^m} \leq \liminf_{\varepsilon \to 0} \|v^\varepsilon(s)\|_{H^m} \leq \frac{\|u(t)\|_{H^m}}{1 - C_1 |s| \|u(t)\|_{H^m}} \text{ for } |s| < \delta .
$$

Hence we indeed have

$$
\limsup_{s \to 0} \|u(t+s)\|_{H^m} \leq \|u(t)\|_{H^m}
$$
as required. \( \square \)
4 Vorticity and Continuation

In this short section our goal is to study the way(s) in which solutions to the Cauchy problem for the Euler equation can fail to be global, i.e. fail to exist for all time. It turns out that, for the Euler equation, the failure of global existence is characterised by the rapid growth of the vorticity. Specifically, we have the following result.

**Theorem 4.1.** Let \( m \in \mathbb{Z} \) with \( m \geq 4 \), and suppose \( u \) is a strong solution to the Euler equation in \( V^m(\mathbb{R}^3) \) up to time \( T^* > 0 \), and that \( u \) cannot be continued beyond time \( T^* \), i.e. there does not exist \( \tilde{u} \in C^0([0,T), V^m(\mathbb{R}^3) \cap C^1([0,T), V^{m-1}(\mathbb{R}^3)) \) with \( T > T^* \), which extends \( u \).

Then necessarily
\[
\int_0^{T^*} \| \omega(t) \|_{L^\infty} \, dt = \infty.
\]
where \( \omega(t) := \text{rot } u(t) \) is the vorticity of the velocity field \( u \).

This section is devoted to the proof of Theorem 4.1. The presentation follows that of [2]. A key tool here is the Biot-Savart law, which allows us to recover the (divergence-free) velocity field from its vorticity. We need the following facts from potential theory.

**Lemma 4.2.** Let \( f : \mathbb{R}^3 \to \mathbb{R} \) be bounded and locally Hölder continuous. Then its Newtonian potential
\[
\Phi[f](x) := -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} \, dy
\]
belongs to \( C^2(\mathbb{R}^3) \) and satisfies
\[
\nabla \Phi[f](x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} f(y) \, dy
\]
and \( \Delta \Phi[f] = f \). □

This result is fairly standard and is a special case of well-known results, so we omit the proof and refer the interested reader to, for example, [5], Lemmas 4.1 and 4.2.

**Proposition 4.3 (Biot-Savart Law).** Let \( \omega \in V^2(\mathbb{R}^3) \). Then there exists a unique \( u \in V^3(\mathbb{R}^3) \) with \( \text{rot } u = \omega \). Moreover we have the exact formula
\[
u(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \wedge \omega(y) \, dy.
\]

**Proof.** We deal with uniqueness first. Consider the Fourier transforms \( \hat{u} \) and \( \hat{\omega} \). The condition \( \text{div } u = 0 \) forces \( \xi \cdot \hat{u}(\xi) = 0 \), and the requirement \( \text{rot } u = \omega \) forces \( i\xi \wedge \hat{u}(\xi) = \hat{\omega}(\xi) \). From this it follows that
\[
\hat{u}(\xi) = i\frac{\xi \wedge \hat{\omega}(\xi)}{|\xi|^2}.
\]

This proves uniqueness. For existence we can simply define \( \hat{u} \) by the preceding formula, and check that it works.

It remains to prove the representation formula. Let \( \Psi : \mathbb{R}^3 \to \mathbb{R}^3 \) be the Newtonian potential of \( \omega \). Since the Sobolev embedding \( H^2(\mathbb{R}^3) \hookrightarrow C^0_{b,1/2}(\mathbb{R}^3) \) holds, Lemma 4.2 applies, and we have \( \text{div } \Psi = 0 \), \( \Delta \Psi = \omega \), and
\[
-\text{rot } \Psi(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \wedge \omega(y) \, dy.
\]
Now \( \Delta \Psi = \nabla(\text{div } \Psi) - \text{rot rot } \Psi \), so we identify the solution \( u = -\text{rot } \Psi \). □
Corollary 4.4. There exists a constant $C > 0$ independent of $u$, such that
$$\|\nabla u\|_{L^2} \leq C \|\text{rot} u\|_{L^2} \quad \text{for every } u \in V^3(\mathbb{R}^3).$$

Proof. From the previous proof, the Fourier transforms of $D_i u_j$ and $\omega := \text{rot} u$ are related by
$$\mathcal{F}[D_i u_j](\xi) = -\xi_i \frac{1}{|\xi|^2} \mathcal{F}[\omega](\xi).$$
That is, $\mathcal{F}[D] = S(\xi)\mathcal{F}[\omega]$ for a matrix-valued function $S : \mathbb{R}^3 \to \mathbb{R}^{3 \times 3}$, which is bounded independently of $\xi \in \mathbb{R}^3$. The result now follows.

Using the formula given by the Biot-Savart law, we have the following estimate, which will be essential to the proof of Theorem 4.1.

Lemma 4.5. There is a constant $C > 0$ such that any $u \in V^3(\mathbb{R}^3)$ satisfies
$$\|\nabla u\|_{L^\infty} \leq C \left[1 + \|\text{rot} u\|_{L^2} + \left(1 + \log^+ \|u\|_{H^1}\right) \|\text{rot} u\|_{L^\infty}\right]$$
where $\log^+ := \max(0, \log)$.

Proof. For convenience denote $\omega := \text{rot} u$. We write the Biot-Savart law as
$$u(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x - y}{|x - y|^4} \cdot \omega(y) \, dy =: \int_{\mathbb{R}^3} K(x - y)\omega(y) \, dy$$
with $K$ a matrix valued function $\mathbb{R}^3 \to \mathbb{R}^{3 \times 3}$.

For $\rho \in (0, 1]$, pick a smooth cut-off function $\zeta_\rho$ such that
$$\zeta_\rho \equiv 1 \text{ on } B(0, \rho), \quad \zeta_\rho \equiv 0 \text{ on } \mathbb{R}^3 \setminus B(0, 2\rho), \quad \|\nabla \zeta_\rho\|_{L^\infty} \leq \frac{C}{\rho}$$
where $C$ is a constant independent of $\rho$. We write
$$\nabla u(x) = \int_{\mathbb{R}^3} K(x - y)\nabla \omega(y) \, dy$$
$$= \int_{\{|x - y| \leq 2\rho\}} \zeta_\rho(x - y)K(x - y)\nabla \omega(y) \, dy$$
$$+ \int_{\{|x - y| < 1\}} \nabla [(1 - \zeta_\rho(x - y))K(x - y)]\omega(y) \, dy$$
$$+ \int_{\{|x - y| \geq 1\}} \nabla [(1 - \zeta_\rho(x - y))K(x - y)]\omega(y) \, dy$$
$$=: I_1 + I_2 + I_3.$$
To estimate $I_3$, note $|\nabla K| \leq C|\cdot|^{-3}$ implies that $|\nabla K| \in L^2(\mathbb{R}^3 \setminus B(0,1))$, so
$$|I_3| \leq C \omega_{L^2}.$$ Putting the above estimates together, we find
$$\|\nabla u\|_{L^\infty} \leq C \left[ \rho^4 \|u\|_{H^3} + \|\omega\|_{L^2}^2 + (1 - \log \rho) \|\omega\|_{L^\infty} \right].$$
Now we choose $\rho$ as follows: If $\|u\|_{H^3} \leq 1$, set $\rho = 1$; otherwise set $\rho = \|u\|_{H^3}^{-4}$. In either case we obtain the asserted bound.

**Proof of Theorem 4.1.** Assume for a contradiction that $T_\ast < \infty$ and yet
$$M_0 := \int_0^{T_\ast} \|\omega(t)\|_{L^\infty} \, dt < \infty.$$ Now the vorticity satisfies
$$\partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u.$$ Since $u$ is divergence-free, $(u \cdot \nabla \omega, \omega)_{L^2} = 0$, so that, using Corollary 4.4,
$$\frac{d}{dt} \left( \|\omega\|_{L^2}^2 \right) = 2 (\omega \cdot \nabla u, \omega)_{L^2} \leq C \|\omega\|_{L^\infty} \|\nabla u\|_{L^2} \|\omega\|_{L^2} \leq C \|\omega\|_{L^\infty} \|\omega\|_{L^2}^2.$$ Thus, for $0 \leq t \leq T_\ast$ we have, by Gronwall’s inequality,
$$\|\omega(t)\|_{L^2} \leq \|\omega(0)\|_{L^2} \exp (CM_0) =: C (M_0, \|u\|_{H^3}) < \infty.$$ In other words, $\|\omega\|_{L^2}$ is uniformly bounded on $[0, T_\ast)$.

By following the proof of Lemma 3.15, it is easy to show
$$\frac{d}{dt} \left( \|u\|_{H^m}^2 \right) \leq C \|\nabla u\|_{L^\infty} \|u\|_{H^m}^2.$$ Using Lemma 4.5, the uniform boundedness of $\|\omega\|_{L^2}$ on $[0, T_\ast)$, and the observation that $1 + \log^+ r \leq 2 \log(e + r)$ for all $r > 0$, we find that there exists a constant $C_1 > 0$, depending on $M_0$ and $\|u(0)\|_{H^3}$, such that
$$\frac{d}{dt} \left( \|u\|_{H^m}^2 \right) \leq C_1 \left[ 1 + \|\omega\|_{L^\infty} \log \left( e + \|u\|_{H^m}^2 \right) \right] \|u\|_{H^m}^2 \leq C_1 \left[ 1 + \|\omega\|_{L^\infty} \log \left( e + \|u\|_{H^m}^2 \right) \right] \left( e + \|u\|_{H^m}^2 \right),$$ that is,
$$\frac{d}{dt} \log \left( e + \|u\|_{H^m}^2 \right) \leq C_1 \left[ 1 + \|\omega\|_{L^\infty} \log \left( e + \|u\|_{H^m}^2 \right) \right].$$ Gronwall’s inequality then gives, for any $t \in [0, T_\ast)$,
$$\log \left( e + \|u(t)\|_{H^m}^2 \right) \leq \log \left( e + \|u(0)\|_{H^m}^2 \right) \exp \left( C_1 \int_0^t \|\omega(\tau)\|_{L^\infty} \, d\tau \right) + C_1 \int_0^t \exp \left( C_1 \int_0^s \|\omega(\tau)\|_{L^\infty} \, d\tau \right) \, ds \leq \left[ \log \left( e + \|u(0)\|_{H^m}^2 \right) \right] + C_1 T_\ast \exp (C_1 M_0) =: C_0.$$ In particular, $\|u\|_{H^m}$ is uniformly bounded in $[0, T_\ast)$.

Hence, for $t \in [0, T_\ast)$ arbitrarily close to $T_\ast$, the local existence theorem (Theorem 3.21) allows us to construct a strong solution starting at $u(t)$ with existence time $T_1(C_0)$ independent of $t$. Therefore we could extend the solution $u$ beyond time $T_\ast$, in contradiction to the definition of $T_\ast$. \qed
References


