The Cauchy problem for the Boltzmann equation

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1 Introduction

New mathematical results do not emerge often from existing tools and techniques. The Boltzmann equation is one of the core equations of mathematical physics, and its Cauchy problem was and still is one of the most important open problems in the field. The new concepts that are key to the 1989 result by DiPerna and Lions [1] are the notions of renormalised solutions of transport equations and velocity averaging. This allows for a proof of the global existence of weak solutions via compactness arguments without any a priori estimates on the derivatives. The regularity and uniqueness of these solutions is still an open problem. As is common when a new method is developed, further results came quickly. In this project we will present the proof of weak solutions to the Boltzmann equation and a later application of the same tools by DiPerna and Lions [3] to ODEs to extend the Cauchy-Lipschitz theorem to some non-Lipschitz forcefields in a slightly weaker sense.

Contents

1 Introduction 1

2 Linear transport equations 2

2.1 Basic overview 2

2.1.1 Characteristics and ‘transportation’ 2

2.1.2 Mild solutions and Duhamel’s principle 3

2.1.3 Algebraic properties and conservation laws 4

2.1.4 Renormalised solutions 4

2.2 The setting 5

2.2.1 Function Spaces and topologies 5

2.2.2 Distributional solutions 6

2.3 Existence and uniqueness 7

2.3.1 Smooth coefficients 8

2.3.2 Rough coefficients 8

2.4 Velocity averaging results 9

2.4.1 Characteristics 9

2.4.2 Velocity Averaging 10

3 The Cauchy Problem for the Boltzmann Equation 13

3.1 The equation 13

3.1.1 Solution concepts for the Boltzmann equation 14

3.1.2 Main theorem 15

3.2 Short sketch of the proof 15

3.3 A priori estimates 16

3.3.1 Entropy identity 18

3.4 Truncated problems 20

3.5 Weak compactness results 24

3.5.1 Weak compactness of \( f^n \) 24

3.5.2 Weak compactness of the collision term 25
2 Linear transport equations

This section will give a refresher on basic linear transport theory, some notions of solution, and then some existence results. To begin with the presentation will be unrigorous, with no spaces or regularity specified. The existence and uniqueness results are sketched from [3], and the equivalence result is combined from [3] and [2].

2.1 Basic overview

The equation we will look at is (with the Einstein summation convention),

\[
\frac{\partial u}{\partial t} + b_i(t,x) \frac{\partial u}{\partial x_i} + c(t,x)u = f(t,x,u)
\]

\[
u(0,x) = u_0(x)\]

\[
u(t,x) : (0,T) \times \mathbb{R}^N \rightarrow \mathbb{R}
\]

To begin with we will look at when \( c = 0 \) and \( f = f(t,x) \).

2.1.1 Characteristics and ‘transportation’

Locally we can view \( b \) as constant, and we see that \( \frac{\partial u}{\partial t} + b_i \frac{\partial u}{\partial x_i} \) is the directional derivative along \((1,b)\). The equation therefore states that \( u \) remains constant along this line. We can then solve the equation by finding the value of \( u \) along this line backwards in time. Globally we need to stitch up all the locally constant lines to form the characteristic curves.

The characteristic equations are

\[
X(s;t,x) : \mathbb{R}^{1+N} \rightarrow \mathbb{R}^N
\]

\[
\dot{X} = b(t,X)
\]

\[
X(s;s,x) = x
\]
where $\dot{X} = \frac{\partial X}{\partial t}$. If $X(t, x)$ solves this and $u$ solves (1), then we observe that for each fixed $s$, $u(t, X(t, x))$ obeys
\[
\frac{d}{dt} [u(t, X(t, x))] = \frac{\partial u}{\partial t}(t, X(t, x)) + \frac{\partial u}{\partial x_i} \frac{\partial X_i}{\partial t} = \left( \frac{\partial u}{\partial t} + b_i \frac{\partial u}{\partial x_i} \right)_{(t, X(t, x))} = 0
\]
Noting that at $t = s = 0$ we have $u(0, X(0; 0, x)) = u(0, x) = u_0(x)$, we see that the value of $u$ at $(t, X(0; t, x))$ is $u_0(x)$. Thus by the group property $(X(s; t + \tau, x) = X(s; t, X(t; \tau, x)))$ and that each $X(s; t, \cdot)$ is a diffeomorphism, we have that if $y = X(0; t, x)$ then $x = X(t; -t, y)$ and hence $u(t, y) = u_0(X(t; -t, y))$. This gives us the solution,
\[
u(t, x) = u_0(X(t; -t, x)) \tag{2}
\]
$X(t; -t, x)$ means we look back along the characteristic curve that starts at $(t, x)$ to find where it crosses the plane $t = 0$. In this way, the transport equation rearranges the function $u(t, \cdot)$ over time, ‘transporting’ mass along the characteristics.

### 2.1.2 Mild solutions and Duhamel’s principle

(3) has meaning for any measurable function $u_0$, so we can extend our notion of solution to a far more general space. To do this we introduce the solution operator $S_t$.

**Definition 1** (Solution Operator). The solution operator to (1) with $c = f = 0$ is an operator on measurable functions $\mathbb{R}^N \rightarrow \mathbb{R}^N$ defined by $$(S_t g)(x) = g(X(t; -t, x))$$

If $b$ depends only on $x$, then $S$ obeys the group property $S_t S_s = S_{t+s}$.

From now on we will write $g^\sharp(t, x)$ to mean $S_t g$, moving our equation along the characteristics. In general this is not a group or even a semigroup due to the time dependence of $b$.

**Definition 2** (Mild solution). The mild solution to $\frac{\partial u}{\partial t} + b_i \frac{\partial u}{\partial x_i} = 0$, $u(0, x) = u_0$ with $u_0$ measurable is $u^\sharp_0$.

All classical solutions are mild solutions.

We now look at when $c$ is not identically zero. Along the characteristics we have
\[
\frac{d}{dt} u^\sharp + c^\sharp u^\sharp = 0
\]
we can solve this via the integrating factor method to give
\[
u^\sharp(t, x) = u^\sharp_0(x) \exp \left[ - \int_0^t c^\sharp(s, x) \, ds \right]
\]
which we can use to define a new solution operator.

**Definition 3** (Solution Operator). The solution operator to (1) with $f = 0$ is defined by $$(S_t g)(x) = g^\sharp(x) \exp \left[ - \int_0^t c^\sharp(s, x) \, ds \right]$$

We will now define $g^\flat$ using this. This is an sort of ‘exponential’ form of the solution. Note that if $c = 0$ then $g^\flat = g^\sharp$.

For the complete transport equation (1), we can use Duhamel’s principle. Formally, along the characteristics we have
\[
\frac{d}{dt} u^\flat = f^\flat
\]
so integrating we obtain
\[
u^\flat(t, x) - u^\flat_0(x) = \int_0^t \frac{d}{dt} u^\flat(s, x) \, dt = \int_0^t f(s, x, u^\flat) \, dt
\]
and we can use this for the general rigorous definition.
Definition 4 (Mild Solutions). Let $u_0$ be measurable, then $u$ is a exponentially mild solution to (1) iff

- For almost all $x$, $\xi^2, f(s, x, u) \in L^1[0, T)$
- For all $t \in [0, T)$ and for almost all $x$,
  \[ u^\flat(t, x) = u_0^\flat(x) + \int_0^t f(s, x, u) \, ds \]

If $c = 0$ we simply call this a mild solution. We can always take $c = 0$ by merging the $cu$ term into $f$, (which can depend on $u$).

2.1.3 Algebraic properties and conservation laws

We now look at the properties of solutions to the transport equation with $f = c = 0$. Suppose $u$ is a solution to the transport equation. A renormalisation of $u$ is given by $\beta(u)$ where $\beta : \mathbb{R} \to \mathbb{R}$. Formally, using the chain rule,

\[ \frac{\partial \beta(u)}{\partial t} + b_i \frac{\partial \beta(u)}{\partial x_i} = \beta'(u) \left( \frac{\partial u}{\partial t} + b_i \frac{\partial u}{\partial x_i} \right) = 0 \]

so that $\beta(u)$ also solves the transport equation with initial condition $\beta(u(0)) = \beta(u_0)$. This works because the equation is linear, first order and has no feedback term as $c = 0$ and $f = 0$. Even when this is not the case, we still obtain that $\beta(u)$ satisfies

\[ \frac{\partial \beta(u)}{\partial t} + b_i \frac{\partial \beta(u)}{\partial x_i} + cu \beta'(u) = f \beta'(u) \]

This makes the next property useful. We look at the evolution of the mass of the solution when $c = f = 0$.

\[ \frac{d}{dt} \int u \, dx = \int \frac{\partial u}{\partial t} \, dx = - \int b_i \frac{\partial u}{\partial x_i} \, dx = \int u \, \text{div} \, b \, dx \]

In particular if $\text{div} \, b = 0$ then the mass is conserved, and we can approximate the $L^p$ norms to show their conservation (for $u_0 \geq 0$). As $\beta(u)$ is also a solution, we have that

\[ \frac{d}{dt} \int \beta(u) \, dx = \int \beta(u) \, \text{div} \, b \, dx \]

These conservation properties can be extended to the full equation where $c, f$ are included. We obtain

\[ \frac{d}{dt} \int \beta(u) \, dx = \int \beta(u) \, \text{div} \, b - cu \beta'(u) - f \beta'(u) \, dx \]

2.1.4 Renormalised solutions

We will now transition the intuition discussed above into rigorous mathematical theorems. Again, we restrict to when $c = f = 0$. When $u$ is very irregular, i.e. not even $L^1_{loc}$, we can take $\beta \in C^1(\mathbb{R})$ bounded and then look at the equation for $\beta(u)$. Of course this equation cannot be solved in the classical sense. Instead we will view it in the sense of distributions. This allows us to define a notion of another solution for very general $u$.

Definition 5 (Renormalised Solution). $u$ is a renormalised solution to (1) (with $c = f = 0$) if and only if for all continuously differentiable $\beta : \mathbb{R} \to \mathbb{R}$, $\beta(u)$ obeys

\[ \frac{\partial \beta(u)}{\partial t} + b_i \frac{\partial \beta(u)}{\partial x_i} = 0 \quad \text{in} \ D'(\mathbb{R}^N) \]

Depending on the space $u$ lies in, we will require different additional restrictions on $\beta$. For example, we may demand that $\beta \in L^\infty$. We will discuss this later.

For the general case, we use the derived equation for $\beta(u)$ and obtain the following definition:
Definition 6 (Renormalised Solution). We say that $u$ is a renormalised solution to \( (1) \) if and only if for all continuously differentiable $\beta : \mathbb{R} \to \mathbb{R}$, with $|\beta'(z)|(1 + |z|)$ bounded, $\beta(u)$ obeys

$$\frac{\partial \beta(u)}{\partial t} + b_i \frac{\partial \beta(u)}{\partial x_i} + cu\beta(u) = f\beta'(u) \text{ in } D'(\mathbb{R}^N)$$

The condition $|\beta'(z)|(1 + |z|)$ bounded means that the extra term $u\beta'(u)$ is bounded also. To define this with a non-zero $f$, we require a restriction on $\beta$ to make $f\beta'(u)$ regular. This will depend heavily on the particular $f$. Later we will look at the particular case of the Boltzmann equation.

The notion of renormalised solutions is particular to the form of the transport equation. It requires the equation to be first order and quasilinear. It allows us to deal with growth of $u$ and $f$ by making the terms in the renormalised equation of lower order.

2.2 The setting

2.2.1 Function Spaces and topologies

We saw before that the transport equation \( (1) \) can be viewed as rearranging $u$ under the forcefield $b$. This means that we can make our notions of solution and spaces very general, even as general as all measurable functions. To do this we need to say what topology we want on these spaces.

Definition 7. We define $L^1$ as the space consisting of measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ that are finite almost everywhere. We give it the topology of local convergence in measure. This has a metric

$$d(\phi, \psi) = \sum_{n \geq 1} \frac{1}{2^n} \|\phi - \psi \wedge 1\|_{L^1(B_n)}$$

This topology is equivalent to the topology induced by the notion of convergence given by $u^n \to u$ if and only if $\beta(u^n) \to \beta(u)$ in $L^1_{loc}$ for all $\beta \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

We have two variables for most of our functions, so we need to define our regularity on each and the way they depend on each other.

Definition 8. We define $L^1([0, T]; L^\infty(\mathbb{R}^N))$ by the norm $\|f\| = \int_{[0, T]} \|u(t, \cdot)\|_\infty \, dt$ and similarly for any other composed spaces with norms.

For spaces without norms, we have to be more careful.

Definition 9. The space $C(\mathbb{R}; L)$ consists of all continuous functions from $\mathbb{R}$ to $L$ where we use the topological definition of continuity. Similarly for other spaces.

Definition 10. $f \in L^\infty(0, T; L)$ means that for all $\beta \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $\beta(f) \in L^\infty(0, T; L^1_{loc})$.

Finally we need to define what we mean by the sum of spaces.

Definition 11. By $f \in L^1 + L^\infty$ we merely mean that $f = f_1 + f_2$ for some $f_1 \in L^1$ and $f_2 \in L^\infty$. This sum will not be unique.

This notation is long winded and distracting. For this reason we will abbreviate. Unless otherwise specified

- Our time interval will be $[0, T]$, and our spacial domain $\mathbb{R}^N$.
- We will assume that the coefficients $b, c, \text{div } b$ will be $L^1$ in time, so that when we say $b \in L^\infty$ we mean $b \in L^1(0, T; L^\infty(\mathbb{R}^N))$. Similarly for convergence of these.
2.2.2 Distributional solutions

Earlier we assumed that it was obvious what is meant for $u$ to satisfy the transport equation in the distributional sense. We will now make this rigorous.

**Definition 12 (Test functions).** We define $\mathcal{D} = \mathcal{D}([0,T] \times \mathbb{R}^N)$ as the space of all $C^\infty([0,T] \times \mathbb{R}^N)$ functions with compact support in $[0,T] \times \mathbb{R}^N$, where by saying that the support is compact in $[0,T)$ we mean that the support can include 0, but must not contain $T$.

**Definition 13 (Distributional Solution).** By saying that $u$ solves \([1]\) in $\mathcal{D}'$ with $f = f(t,x)$ we mean that for all $\phi \in \mathcal{D}$,

$$
\int_0^T \int_{\mathbb{R}^N} u \left\{ \frac{\partial \phi}{\partial t} - \text{div}(b\phi) + c\phi \right\} dx \, dt - \int_0^T \int_{\mathbb{R}^N} u_0 \phi(0,x) dx = \int_0^T \int_{\mathbb{R}^N} f(t,x)\phi \, dx \, dt
$$

This makes sense for $u \in L^p$ if we have the conditions on the coefficients

$$
c - \text{div} b \in L^q_{\text{loc}}, \quad b \in L^q_{\text{loc}}, \quad f(t,x) \in L^1_{\text{loc}}
$$

where $p, q$ are Hölder conjugates.

In particular for $u \in L^\infty$, we require that $c - \text{div} b, b$ and $f$ are in $L^1_{\text{loc}}$.

In a broad sense these notions of solutions are equivalent. In particular:

**Theorem 1 (Equivalence).** Let the $\sharp$ operator be invertible and measure preserving (for this to hold it is sufficient that $\text{div} b = 0$, $b$ is independent of $t$ and satisfies the upper bound $|b(z)| \leq C(1 + |z|)$). Then:

- If $u, f \in L^1_{\text{loc}}([0,T] \times \mathbb{R}^N)$, then mild and distributional solutions are equivalent.
- If $f\beta'(u) \in L^1_{\text{loc}}([0,T] \times \mathbb{R}^N)$ then mild solutions are renormalised solutions.
- Renormalised solutions are mild solutions, and we need only check $\beta(z) = \log(1 + z)$.
- If in addition $c \geq 0$ and for almost all $x$, $(ca)^2 \in L^1[0,T)$, then exponentially mild solutions are mild solutions.

**Proof.** We start with the equivalence of mild and distributional solutions. Here we will merge $c$ into $f$, so wlog we may assume that $c = 0$.

Let $u$ be a distributional solution, and fix a test function $\phi \in \mathcal{D}(\mathbb{R}^N)$. Let $\rho_n \in \mathcal{D}([0,T])$ satisfy $\rho_n \to \mathbf{1}_{[t_1,t_2]}$ in $\mathcal{D}'([0,T])$. By measure preservation and invertibility the adjoint of $\sharp$ is given by $\sharp(-t,x)$ and both operate on all measurable functions, so that when we test the equation with $\phi^2(-t,x)\rho_n(t)$ we obtain:

$$
0 = \left\langle \frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x_i} - f, \phi^2(-t,x)\rho_n(t) \right\rangle
\quad = - \left\langle u, \left[ \frac{\partial}{\partial t} + b_i \frac{\partial}{\partial x_i} \right] (\phi^2 (-t,x)\rho_n(t)) \right\rangle - \left\langle f, \phi^2(-t,x)\rho_n(t) \right\rangle
\quad = - \left\langle u, \rho_n(t) \left[ \frac{\partial}{\partial t} + b_i \frac{\partial}{\partial x_i} \right] \phi^2(-t,x) + \phi^2(-t,x)\rho_n'(t) \right\rangle - \left\langle f, \phi^2(-t,x)\rho_n(t) \right\rangle
\quad = - \left\langle u, \phi^2(-t,x)\rho_n'(t) \right\rangle - \left\langle f, \phi^2(-t,x)\rho_n(t) \right\rangle
$$

Where the inner product is the action of distributions in $\mathcal{D}'([0,T] \times \mathbb{R}^N)$ and we note that the equality $\left[ \frac{\partial}{\partial t} + b_i \frac{\partial}{\partial x_i} \right] \phi^2(-t,x) = 0$ follows from the chain rule (as $\phi$ is smooth) and the definition of $\sharp$. As $\phi$ was arbitrary, we have $\int_0^T \rho_n'(t)f^2 - \rho_n(t) u^2 \, dt = 0$ for almost all $x$. We then let $n \to \infty$ to obtain that for almost all $x$,

$$
u^2(t_2,x) - u^2(t_1,x) = \int_{t_1}^{t_2} f^2(s,x) \, ds
$$
So $u$ is a mild solution. The converse comes from reversing this argument.

Now let $u$ be a renormalised solution. Define $\beta(z) = \log(1 + z)$. By the definition of renormalised solutions
\[
\frac{\partial}{\partial t} + b \frac{\partial}{\partial x_i} \beta(u) = \beta'(u)f = \frac{f}{1 + u}
\]
in $\mathcal{D}'$

Next, define $\beta_\delta(z) = \frac{1}{\delta}(\delta(e^z - 1))$ for $\delta > 0$, which, when $\delta \to 0$, will approximate the identity function. Note that for fixed $\delta$ each $\beta_\delta$ is Lipschitz. We then have
\[
\frac{\partial}{\partial t} + b \frac{\partial}{\partial x_i} \beta_\delta(u) = \beta_\delta'(u)f = \frac{f}{1 + u}
\]
in $\mathcal{D}'$

But we see that $\beta_\delta'(\beta(z)) = e^{\beta(z)}\beta'(\delta e^{\beta(z)} - 1) = (1 + z)\beta'(\delta z) = \frac{1 + z}{1 + \delta z}$, so that for all $\delta > 0$,
\[
\frac{\partial}{\partial t} + b \frac{\partial}{\partial x_i} \beta_\delta(u) = \frac{f}{1 + u}
\]
in $\mathcal{D}'$

By computation we see that $v_\delta = \beta_\delta(u) = \frac{1}{\delta}\log(1 + \delta u)$, and by the equivalence of mild and distributional solutions proved above, we have that for almost all $x$, $f^\delta \in \mathcal{L}^1[0, T]$. We now go back to $u$ by noting that $u^\delta = e^{v_\delta} - 1$, so that $u^\delta$ is also the time integral of some function, and hence $f^\delta$, (which may depend on $u$), is in $\mathcal{L}^1[0, T]$ as required. Then by the equivalence again we obtain that for all $t > s \geq 0$, $\delta > 0$ and almost all $x$,
\[
v_\delta^s(t, x) - v_\delta^s(s, x) = \int_s^t \frac{f^\delta}{1 + \delta u^\delta} \, d\tau
\]
When we take $\delta \to 0$ we see that $u$ is a mild solution.

The converse follows from putting observing that if $u$ is a mild solution then $v_1^s$ obeys the equation above, and we then apply the equivalence of mild and distributional solutions.

Finally we have to show that exponentially mild solutions are equivalent to mild solutions. Suppose $u$ is an exponentially mild solution and the given conditions on $c$ are satisfied. Then we have

- For almost all $x$, $c^\delta, f^\delta \in \mathcal{L}^1[0, T]$.
- For $0 \leq s < t < T$ and almost all $x$, $w^\delta(t, x) - w^\delta(s, x) = \int_s^t f(\tau, x, u^\delta) \, d\tau$.
- $c \geq 0$
- For almost all $x$, $(cu)^\delta \in \mathcal{L}^1[0, T]$

and we need to show that

- For almost all $x$, $(f - cu)^\delta \in \mathcal{L}^1[0, T]$.
- For $0 \leq s < t < T$ and almost all $x$, $w^\delta(t, x) - w^\delta(s, x) = \int_s^t (f - cu)^\delta(\tau, x) \, d\tau$

Recall that $f^\delta = f^\delta \exp\left(-\int_0^s c^\delta(s, x) \, ds\right)$, and so because $c \geq 0$ and $f^\delta \in \mathcal{L}^1[0, t]$ we see that $f^\delta \in \mathcal{L}^1[0, T]$, and hence $(f - cu)^\delta \in \mathcal{L}^1[0, T]$. Under these conditions we can go from $w^\delta$ to $u^\delta$ and obtain a mild solution. \hfill \Box

### 2.3 Existence and uniqueness

We will show existence and uniqueness with initial conditions
\[
u_0 \in \mathcal{L}^p, \quad p \in [1, \infty]
\]
and with coefficients that satisfy
\begin{align}
c, \text{div } b &\in \mathcal{L}^1(\mathcal{L}^\infty) \\
b &\in \mathcal{L}^1(\mathcal{W}^{1,q}_{\text{loc}}) \\
\frac{b}{1 + |x|} &\in \mathcal{L}^1(\mathcal{L}^1) + \mathcal{L}^1(\mathcal{L}^\infty)
\end{align}
2.3.1 Smooth coefficients

**Lemma 1.** Assume (3) and (4), and let \( b(t, x), c(t, x), f(t, x) \in C^\infty \). Then there is a classical smooth solution.

**Proof.** By Cauchy-Lipschitz there is a unique and smooth (in both \( x \) and \( t \)) solution to the characteristic equations, and by the growth condition on \( b \) it is global. We can then define our solution by the solution operators defined previously, all of which is valid for in the smooth case.

2.3.2 Rough coefficients

The estimates given in 2.1.3 together with the previous smooth result and standard mollification arguments yield:

**Proposition 2** (Existence in \( L^p \)). Assume (3) and (4) are satisfied. Then there is a unique distributional solution in \( C(0, T; L^p) \) or if \( p = \infty \) merely \( L^\infty(0, T; L^\infty) \).

**Proof.** Mollification allows the application of the smooth result. Using the ideas of 2.1.3 and approximation we obtain the estimates

\[
\| u(t) \|_\infty \leq \| u_0 \|_\infty + \int_0^t \| cu \|_\infty \, ds \leq C \| u_0 \|_\infty
\]

\[
\frac{d}{dt} \int |u(t)|^p \, dx \leq \left( \int |u(t)|^p \, dx \right) \| pc + \text{div } b \|_\infty
\]

\[
\| u(t) \|_p \leq C' \| u_0 \|_p \text{ a.e. on } (0, T)
\]

by Gronwall.

We can now apply relative weak compactness criterion to see that our sequence of solutions \( (\tilde{u}) \) is relatively weak compact if \( 1 < p < \infty \) and relatively weak* compact for \( p = \infty \). A computation that we skip then shows that a limit of a convergent subsequence is a solution.

Finally we must show uniqueness. To do this we will make the conservation laws developed above rigorous. This requires a technical lemma:

**Lemma 2.** Under the conditions of the theorem, the mollification \( \tilde{u} = \tilde{u}(\epsilon; t, x) \) of the actual solution (not the solution to the equation with mollified coefficients) satisfies

\[
\frac{\partial \tilde{u}}{\partial t} + b_i \frac{\partial \tilde{u}}{\partial x_i} + c \tilde{u} = \tilde{r}
\]

where as \( \epsilon \to 0 \), \( \tilde{r} \to 0 \) in \( L^1(L^p) \) if \( p < \infty \) or in \( L^1(L^{p'}) \) all \( p' < \infty \) for \( p = \infty \).

The only hard part of the proof of this is showing that the error of mollifying before taking \( \frac{\partial}{\partial x_i} \) instead of after in the term involving \( b \) tends to 0. We omit this.

We wish to deduce the equation for \( \beta(u) \). Let \( \tilde{u} \) be the mollification of the solution. By the lemma \( \tilde{u} \) is the smooth solution to an equation. Put \( \beta(u) \) in this for \( \beta \in C^1, \beta' \) bounded. Then we get

\[
\frac{\partial \beta(\tilde{u})}{\partial t} + b_i \frac{\partial \beta(\tilde{u})}{\partial x_i} + c \beta(\tilde{u}) = \tilde{r} \beta'(\tilde{u})
\]

Taking \( \epsilon \to 0 \) the right hand side tends to zero, and we get

\[
\frac{\partial \beta(u)}{\partial t} + b_i \frac{\partial \beta(u)}{\partial x_i} + c \beta(u) = 0 \quad \text{in } D'
\]

For \( p < \infty \), approximation arguments using \( \beta \) now give that if \( u_0 = 0 \) then \( u = 0 \) which by linearity gives uniqueness. We will skip most of this, noting only the method.

- Approximate the Lipschitz function \( \beta(z) = \max(|z|, M)^p, M \in (0, \infty) \) using \( C^1 \) functions.
Integrate the equation for $\beta(u)$ against a test function $\phi_R = \phi(\cdot/R)$ where $\phi$ is 1 in $|x| \leq 1$ and 0 outside $|x| \geq 2$. The purpose is to look at $\frac{d}{dt} \int \beta(u) \phi_R \, dx$ and close by Gronwall.

$$\frac{d}{dt} \int \beta(u) \phi_R \, dx = - \int (cu \beta'(u) - \text{div} b \beta(u)) \phi_R \, dx + \int \beta(u) b_i \frac{\partial \phi_R}{\partial x_i} \, dx$$

The first integral is ok because of the boundedness of $\beta'$ and the conditions on the coefficients.

The only problem is the second integral, for which we apply the growth condition $b/|x| \in L^1 + L^\infty$. The problem integral is only non-zero in $|x| \in (R, 2R)$ and we get $1/R$ decay from the derivative on $\phi_R$. In this region we bound $|b| \leq |x|^{-1}$ and $|x|^{-1} \leq 1$ which we split this into $L^1$ and $L^\infty$ parts. The $L^1$ part is integrable when we bound $\beta(u) \leq M$, while the $L^\infty$ part just adds a term to the bound for the first integral.

Sending $R \to \infty$ we can close by Gronwall in the case $p < \infty$.

For $p = \infty$ a duality argument is required involving solving the backwards problem

$$\frac{\partial \Phi}{\partial t} + b_i \frac{\partial \Phi}{\partial x_i} - (c - \text{div} b) \Phi = \varphi$$

for $\Phi(t = 0) = 0$ and test functions $\varphi$. An argument we omit then uses this to show that $\int u \varphi \, dx \, dt = 0$ and the uniqueness.

We have in fact shown that

$$\frac{d}{dt} \|u(t)\|_p^p + \int (pc - \text{div} b)|u|^p \, dx = 0$$

which gives $\|u(t)\|_p \in C([0,T])$. If $p > 1$ then $u \in C(0,T; L^p)$ follows from the form of the transport equation. $p = 1$ is harder, using approximation arguments we omit here. 

## 2.4 Velocity averaging results

The aim of this section is to establish results for the later existence proof for the Boltzmann equation, which has the form

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} = g$$

$$f(t, x, v) : [0,T) \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$$

We will later see the exact form of the right hand side $g = Q(f, f)$, but for now it doesn’t matter. The equation is often written using the transport operator $T = \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i}$ as $ Tf = g$. This is a time independent inhomogeneous transport equation with $b(x,v) = [v, 0]^T$. This means that $\text{div} b = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0$, and the homogeneous equation preserves phase-space. In this section we will look at the properties of equations of the form

$$Tf = g$$

### 2.4.1 Characteristics

The characteristics split into two vectors and the equations have a simple form:

$$\dot{X}(t, x, v) = -V(t, x, v)$$

$$\dot{V}(t, x, v) = 0$$

$$X(0, x, v) = x$$

$$V(0, x, v) = -v$$

This has a solution given by $X(t, x, v) = x - vt$, $V(t, x, v) = v$. The characteristic map is then

$$f^t(t, x, v) = f(t, x + vt, v)$$

from which we gain our notion of mild solution.
2.4.2 Velocity Averaging

Here we will present one of a general class of results called velocity averaging lemmas. The transport operator $T$ gives no gain in regularity, but if we integrate through, we obtain conservation of mass we get a very large gain in regularity. What happens if we only partially integrate through?

**Definition 14** (Velocity average). We define the velocity average of $f(t,x,v)$ by $F(t,x) = \int_{\mathbb{R}^N} f(t,x,v) \, dv$.

We will look at the Fourier transform of $Tf$ in the $t \to \tau$ and $x \to y$ coordinates. In Fourier space the transport operator takes the form $\hat{T} = i\tau + iv_jy_j$, which is very similar to the Sobolev norm $\|f\|_{H^s}^2 = \int_{\mathbb{R} \times \mathbb{R}^N} (1 + |\tau|^2 + |y|^2)^s |\hat{f}(\tau, y, v)|^2 \, d\tau \, dy \, dv$. We can use this to show a gain in regularity from $f$ to $F$.

**Lemma 3.** Let $f \in L^2(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N)$ with compact support, and $Tf \in L^2(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N)$, then $F(t,x) \in H^{1/2}(\mathbb{R} \times \mathbb{R}^N)$.

**Proof.** We wish to bound $\|F\|_{H^{1/2}}$. We compute

$$
\hat{F}(\tau, y) = \int_{\mathbb{R}^N} \hat{f}(\tau, y, v) \, dv
$$

So as $f \in L^2(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N)$ it is sufficient to show that $\int_{\mathbb{R} \times \mathbb{R}^N} (|\tau|^2 + |y|^2)^{1/2} |\hat{f}(\tau, y)|^2 \, d\tau \, dy \, dv$ is bounded. Note that $\int_{\mathbb{R} \times \mathbb{R}^N} |\hat{f}(\tau, y)|^2 \, d\tau \, dy < \infty$. Let $\rho = (|\tau|^2 + |y|^2)^{1/2}$. We will show that $|\hat{F}|^2 \leq \frac{C}{\rho} \left( \|f\|_{L^2}^2 + \|\hat{T}f\|_{L^2}^2 \right)$ and the result will follow as by Jensen’s inequality

$$
\int_{\mathbb{R} \times \mathbb{R}^N} \|u\|_{L^2}^2 \, dx \, dt \leq \int_{\mathbb{R} \times \mathbb{R} \times \mathbb{R}^N} |u|^2 \, dv \, dx \, dt
$$

Note that $|\hat{T}f|^2 = |\tau + v_jy_j|^2 |\hat{f}|^2$. We want to use this to obtain $\frac{1}{\rho}$. To do this, we split the integral according to whether the velocity makes this small or large. Define $\tau^0 = \frac{\tau}{\rho}$ and $y_j^0 = \frac{y_j}{\rho}$. Then $|\hat{T}| = |\tau + v_jy_j| = \rho |\tau^0 + v_jy_j^0|$. For $|\hat{T}| \geq 1$, if $K$ is the compact support of $f$ in $v$, we have

$$
|\hat{F}_{\text{large}}|^2 = \left( \int_{|\tau| \geq 1} |\hat{f}|^2 \, dv \right)^2
\leq \left( \int_{|\tau| \geq 1, \tau \in K} |\hat{T}|^{-2} \, dv \right) \left( \int_{\mathbb{R}^N} |\hat{T}^0|^2 |\hat{f}|^2 \, dv \right)
\leq \frac{1}{\rho^2} \left( \int_{|\tau| \geq 1, \tau \in K} |\tau^0 + v_jy_j^0|^{-2} \, dv \right) \cdot \|\hat{f}\|_{L^2}^2
\leq \frac{1}{\rho^2} \left( \int_{|\tau^0 + v_jy_j^0| \geq 1/\rho, \tau \in K} |\tau^0 + v_jy_j^0|^{-2} \, dv \right) \cdot \|\hat{f}\|_{L^2}^2
\leq \frac{C}{\rho} \|\hat{f}\|_{L^2}^2
$$

by direct integration

While for $|\hat{T}| < 1$ we obtain:

$$
|\hat{F}_{\text{small}}|^2 = \left( \int_{|\tau| < 1} |\hat{f}|^2 \, dv \right)^2
\leq \left( \int_{\tau \in K, |\hat{T}| < 1} 1 \, dv \right) \left( \int_{\mathbb{R}^N} |\hat{f}|^2 \, dv \right)
\leq \frac{C'}{\rho} \|\hat{f}\|_{L^2}^2
$$

where in the last line we take the supremum over $\tau^0, y_j^0$ of the Lebesgue measure of $\{v \in K : |\hat{T}| < 1\} = \{v \in K : |\tau^0 + v_jy_j^0| < 1/\rho\}$ which is a slice of width $\frac{2}{\rho}$ in $\mathbb{R}^N$, intersected with $K$. \hfill \Box
The compactness of $H^{1/2}$ in $L^2$ allows the conversion of weak compactness into strong compactness. The following lemma and corollary will later be used on the Boltzmann equation.

**Lemma 4.** Let the following assumptions hold:

\[(f_n) \subset L^1((0, T) \times \mathbb{R}^N \times \mathbb{R}^N) \text{ be weakly relatively compact.} \tag{6}\]

\[(\psi_n) \text{ be bounded in } L^\infty((0, T) \times \mathbb{R}^N \times \mathbb{R}^N) \text{ and converge a.e. to } \psi.\]

Additionally, assume that \((Tf_n)\) be weakly relatively compact in \(L^1_{\text{loc}}((0, T) \times \mathbb{R}^N \times \mathbb{R}^N)\). Then the velocity average \(\int_{\mathbb{R}^N} f_n \psi_n \, dv\) is strongly compact in \(L^1((0, T) \times \mathbb{R}^N)\).

**Corollary 1.** Furthermore, if in addition \(f_n \to f\) weakly in \(L^1((0, T) \times \mathbb{R}^N \times \mathbb{R}^N)\) then the velocity average converges strongly to \(\int_{\mathbb{R}^N} f \psi \, dv\) in \(L^1((0, T) \times \mathbb{R}^N)\).

**Proof.** By Egorov’s theorem have \(\psi_n \to \psi\) uniformly except on a set of arbitrarily small measure, as we only care about \(L^1\) convergence, we can take \(\psi_n \to \psi\) uniformly everywhere and in fact we can take \(\psi_n = \psi\). In a similar way, by uniform integrability (as \((f_n)\) is relatively weakly compact), for any \(\epsilon > 0\) we can take a fixed compact set \(K\) on whose complement all the integrals are \(< \epsilon\) in magnitude. For this reason we may assume that \((f_n)\) and \(f\) have fixed compact support \(K\). We can also approximate \(\psi \in L^\infty\) by smooth functions \(\psi_k\) uniformly bounded in \(L^\infty\), so that \(f_n \psi_n\) satisfies the same conditions as \(f_n\). This means we may take \(\psi = 1\), as the error is

\[
\int_{\mathbb{R}^N \times \mathbb{R}^N} |\psi_k f_n - f_n| \, dv \, dx \, dt \leq \int_{\mathbb{R}^N \times \mathbb{R}^N} |\psi_k - \psi| |f_n| \, dv \, dx \, dt \to 0
\]

because \((g_n)\) is uniformly integrable. Finally by the compact support of \((f_n)\), we have that \(Tf_n\) is weakly relatively compact in the whole space \(L^1\) rather than simply \(L^1_{\text{loc}}\).

In summary, we now have \((f_n)\) compactly supported and weakly relatively compact in \(L^1\), with \(Tf_n\) weakly relatively compact in \(L^1\), and we want to show \((F_n)\) strongly compact in \(L^2\). The result would then follow from the velocity averaging lemma as \(H^{1/2} \subset\subset H^0 = L^2\) by Rellich-Kondrachov, and because of the compact support, the compactness in \(L^1\).

To do this we need to approximate with \(L^2\) functions. We define the large and small parts of \(f_n\) by the solutions to:

\[T^\text{small}_n = (Tf_n) 1\{|Tf_n| \leq M\}\]
\[T^\text{large}_n = (Tf_n) 1\{|Tf_n| > M\}\]
\[f^\text{large}_n|_{t=0} = f^\text{small}_n|_{t=0} = 0\]

By linearity of \(T\) and uniqueness of solutions, \(f_n = f^\text{small}_n + f^\text{large}_n\). We can then use the explicit solution operator to express \(f^\text{large}_n\) as

\[f^\text{large}_n = \int_0^t (Tg)^2 1\{|Tf_n| > M\} \, ds\]

Then integrating and using measure preservation of the solution operator, we see that uniformly in \(n\)

\[
\|f^\text{large}_n\|_{L^1} \to 0 \quad \text{as } M \to \infty
\]

However, by measure preservation again, both \((f^\text{small}_n)\) and \((Tf^\text{small}_n)\) are bounded in \(L^2\), which is what we needed for the velocity averaging lemma.

**Lemma 5.** Assume that the following hold

\[f_n \to f \text{ weakly in } L^1_+((0, T) \times \mathbb{R}^N \times \mathbb{R}^N)\]

**Proof.** By Rellich-Kondrachov, and because \((f_n)\) is relatively weakly compact. For each fixed \(\delta > 0\), let \((T\beta_\delta(f^n))\) be weakly relatively compact in \(L^1_{\text{loc}}((0, T) \times \mathbb{R}^N \times \mathbb{R}^N)\) \(\tag{7}\)

where \((\beta_\delta)_{\delta > 0}\) is a uniformly Lipschitz family in \(C(\mathbb{R}; \mathbb{R})\) with \(\beta_\delta(0) = 0\)

and \(\beta_\delta(z) \to z\) as \(\delta \to 0\) uniformly on compact sets.

Additionally, assume that \((6)\) holds. Then \(\int_{\mathbb{R}^N} f_n \psi_n \, dv \to \int_{\mathbb{R}^N} f \psi \, dv\).
We then take \( k \psi \) the convergence of \( u \) we have the \( k \)-independent estimate

\[
\|f^n - \beta_\delta(f^n)\|_{L^1} = \int 1\{|f^n| < a\}|f^n - \beta_\delta(f^n)| + \int 1\{|f^n| \geq a\}|f^n - \beta_\delta(f^n)|
\]

The first term tends to 0 by the uniform convergence on compact sets. Using the bound \( \beta_\delta(f^n) \leq M|f^n| \) where \( M \) is the uniform Lipschitz constant, we see that the second integral is less than \((1 + M) \int 1\{|f^n| \geq a\}|f^n|\) which tends to zero by the dominated convergence theorem. Hence we have that \( f^n - \beta_\delta(f^n) \to 0 \) in \( L^1 \).

Now we apply lemma 3 to \((\beta_\delta(f^n))\) for each fixed \( \delta > 0 \), and obtain the result using the triangle inequality, taking \( \delta \to 0 \).

We can also deduce vector valued versions of these results, but first we need a technical lemma.

**Lemma 6.** Let

- \((E, \nu)\) be an arbitrary measure space.
- \( U \subset \mathbb{R}^N \).
- \((u_n)\) be weakly compact in \( L^1(K)\) for any compact \( K \subset U \) with \( \text{supp} \psi \subset K \times E \).
- \((\psi_k)\) be uniformly bounded in \( L^\infty(U; L^1(E))\) and converge to 0 in \( L^1(U \times E)\).

Then \( \sup_n\|u_n\psi_k\|_{L^1(U \times E)} \to 0 \) as \( k \to \infty \).

**Proof.** First it should be noted that the reason \((E, \nu)\) is so general is that we need nothing from it.

Let \( K_m \) be a chain of compact sets in \( U \) that eventually surround any point. Then we note that by the convergence of \( \psi_k \) we have the \( k \)-independent estimate

\[
\sup_n\|\psi_k u_n\|_{L^1(K_m \times E)} \leq C \sup_n\|u_n\|_{L^1(K_m)}
\]

We then bound the large part of \( u^n \),

\[
\|\psi_k u_n 1\{|u_n| \geq M\}\|_{L^1(K_m \times E)} \leq C \|u_n 1\{|u_n| \geq M\}\|_{L^1(K_m)} \leq \gamma(M)
\]

where by the weak compactness of \((u_n)\), \( \gamma(M) \to 0 \) as \( M \to \infty \).

The small part of \( u_n \) can then be bounded by

\[
\|u_n \psi_k 1\{|u_n| < M\}\|_{L^1(K_m \times E)} \leq M \|\psi_k\|_{L^1(K_m \times E)}
\]

We then take \( k \to \infty \) and then \( M \to \infty \) to obtain the result. \( \square \)

**Corollary 2.** Let (3) and (4) hold but with \( \psi_n, \psi \) and their convergence in \( L^\infty((0, T) \times \mathbb{R}^N \times \mathbb{R}^N; L^1(\mathbb{R}^N))\).

Then the velocity average converges,

\[
\int_{\mathbb{R}^N} f_n \psi_n \, dv \to \int_{\mathbb{R}^N} f \psi \, dv \quad \text{strongly in } L^1((0, T) \times \mathbb{R}^N \times \mathbb{R}^N)
\]

**Proof.** By arguments given in prior proofs we can reduce to when \( \psi_n = \psi \). Now \( \psi(t, x, v) \) is valued in \( L^1(\mathbb{R}^N) \). To keep this space distinct we will label it \( E \). If \( \psi \) separates into a sum

\[
\psi(t, x, v) = \sum_{i=1}^m \phi_i(t, x, v) \varphi_i \quad \text{where } \phi_i \in L^1((0, T) \times \mathbb{R}^N \times \mathbb{R}^N) \text{ and } \varphi_i \in L^1(E)
\]

then the result is immediate by lemma 5. Otherwise we can approximate \( \psi \) with functions of this form and apply the technical lemma. \( \square \)
3 The Cauchy Problem for the Boltzmann Equation

3.1 The equation

In this section, we turn to the discussion of the Cauchy problem for the Boltzmann equation, namely

\[
\begin{cases}
\frac{\partial f}{\partial t}(t, x, v) + v \cdot \nabla_x f(t, x, v) = Q(f, f)(t, x, v) \\
f(0, x, v) = f_0(x, v)
\end{cases}
\]

(B)

where \((t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d\). This equation was first formulated in 1872 by Ludwig Boltzmann to model the time evolution of the molecular density of a rarefied gas. Note that if the gas is so rarified that the molecules can be assumed not to interact with each other, then we expect their density to follow the linear transport equation

\[
\frac{\partial f}{\partial t} + v \cdot \nabla_x f = 0,
\]

and hence to remain constant along each molecular path. In this case the problem can be solved explicitly just by evolving the initial datum along characteristics. If, on the other hand, the molecules interact with each other, this conservation no more holds, and a term to describe the change of the molecular density due to these interactions needs to be added to the equation. This corresponds to \(Q(f, f)\) in (B).

In order to explain the structure of \(Q(f, f)\) we need to choose a model for the molecular interaction. In this notes, we focus on the so called hard spheres model, according to which the molecules are described as perfectly elastic spheres moving independently (following the laws of classical mechanics) until they collide. Call \(v\) and \(v_*\) the velocities of two molecules colliding at time \(t\) in position \(x\), and \(v', v'_*\) their velocities after the collision respectively. Since we are assuming the collisions to be perfectly elastic, the momentum and kinetic energy must be conserved, and we obtain the first relations

\[
\begin{align*}
    v + v_* &= v' + v'_* \\
    |v|^2 + |v_*|^2 &= |v'|^2 + |v'_*|^2
\end{align*}
\]

(8)

From this, if we let \(\omega\) be the unit vector, directed along the line joining the centres of the spheres, we deduce

\[
\begin{cases}
    v' = v - \omega[(v - v_*) \cdot \omega] \\
    v'_* = v_* + \omega[(v - v_*) \cdot \omega]
\end{cases}
\]

(9)

and this easily implies the relations

\[
\begin{align*}
    v \cdot v_* &= v' \cdot v'_* \\
    |v - v_*| &= |v' - v'_*| \\
    (v - v_*) \cdot \omega &= -(v' - v'_*) \cdot \omega
\end{align*}
\]

(10)

that will be useful later on. Equations (9) allow us to explicitly define the collision operator \(Q(f, f)\) as the average change in the molecular densities of colliding particles, weighted with the collision kernel.

\[
Q(f, f) = \int_{\mathbb{R}^d} \int_{S^{d-1}} q(v - v_*, \omega)(f' f'_* - f f_*) dv_* d\omega
\]

13
where \( f = f(t, x, v), f_s = f(t, x, v_s), f' = f(t, x, v'), f'_s = f(t, x, v'_s) \), and the hard spheres collision kernel \( q \) is given by
\[
q(v - v, \omega) = |(v - v_s) \cdot \omega|.
\]
Observe that we can interpret \( q \) as the probability of a collision happening between two molecules having velocities \( v, v_s \), and such that the unit vector joining their centres is \( \omega \). The collision operator \( Q \) is the main problem with trying to solve the Boltzmann equation. It is non-linear and also non-local in the sense that its value at a point depends on the values of \( f \) everywhere. Despite this, \( Q \) splits nicely into positive and negative parts, and the negative part can be split into local and non-local parts.

\[
Q_+(f, f) = \int \int f'_s f' q(v - v_s, \omega) dv_s d\omega \\
Q_-(f, f) = f \cdot L(f) \\
L(f) = \int \int q(v - v_s, \omega) f_s dv_s = \int A(v - v_s) f_s dv_s = A \ast f \\
A(z) = \int q(z, \omega) d\omega.
\]

Observe that \( Q_- \) has a local part \( f \) and a non-local part \( A \ast f \). The non-local part will be the main angle of attack, once we have a bound on it, we can use an entropy inequality to bound \( Q_+ \). In this way we control the nasty non-local part of \( Q_+ \) with the nicer but still non-local part of \( Q_- \).

We remark here that more general collision kernels can be considered, and in fact Assumptions \[\text{[1]}\] below are enough to prove the main existence result, namely Theorem \[\text{[5]}\].

### 3.1.1 Solution concepts for the Boltzmann equation

The concepts of mild and renormalised solutions for the Boltzmann equation read as follows:

**Definition 15** (Mild solution of the Boltzmann equation). A mild solution to the Boltzmann equation \[\text{[B]}\] is a measurable function \( f(t, x, v) : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \) for which

- \( Q_+(f, f) \in L^1_{\text{loc}}[0, \infty) \) for almost all \( x, v \), (so the integral below makes sense).
- For all \( t \geq 0 \) and for almost all \( x, v \), we have \( f'(t, x, v) = f_0(x, v) + \int_0^t Q(f, f) \beta(s, x, v) ds \).

**Definition 16** (Renormalised solution of the Boltzmann equation). A renormalised solution to the Boltzmann equation is a measurable function \( f(t, x, v) \in L^1_{\text{loc}}([0, \infty)_{\text{loc}} \times \mathbb{R}^d \times \mathbb{R}^d) \), for which

\[
\frac{Q_+(f, f)}{1 + f} \in L^1_{\text{loc}}([0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d)
\]

(11) (so that it exists as a distribution).

- For all Lipschitz continuous functions \( \beta(z) : [0, \infty) \rightarrow \mathbb{R} \) for which \( |\beta'(z)|(1 + |z|) \in L^\infty \), \( f \) satisfies
\[
T[\beta(f)] = \beta'(f) Q(f, f) \quad \text{in } \mathcal{D}'
\]

Note how the bound on the derivative of \( \beta \) combined with the condition on \( Q \) acts to put the right hand side of the equation in \( L^1_{\text{loc}} \).

**Lemma 7** (Equivalence for Boltzmann equation). The two notions of solutions are equivalent in the following sense.

- If \( f \) is a renormalised solution, but for only \( \beta(z) = \log(1 + z) \), then \( f \) is a mild solution.
- If \( f \) is a mild solution, and \[\text{[1]}\] holds, then \( f \) is a renormalised solution.

**Proof.** We just apply Theorem \[\text{[1]}\] \( \square \)
3.1.2 Main theorem

In order to state the main existence result for (B) we need to make the following assumptions.

Assumptions 1. Let 

\[
\begin{align*}
g &= v - v_* 
\end{align*}
\]

Assume for the collision kernel \( q(z,\omega) \) the following properties:

- \( q \) depends only on \( |z| \) and \( |z \cdot \omega| \)
- \[ \frac{1}{1 + |v|^2} \int_{|v_*| \leq R} A(v - v_*)dv_* \rightarrow 0 \quad \text{as} \quad |v| \rightarrow \infty \] (12)
- \( A \in L_{loc}^\infty(\mathbb{R}^d) \).

Note that the hard spheres kernel \( q(v - v_* , \omega) = |(v - v_*) \cdot \omega| \) trivially satisfies all the above properties.

We also make the following assumptions on the initial datum.

Assumptions 2. Assume the initial datum \( f_0 \) of (B) belongs to \( L_1^1(\mathbb{R}^d \times \mathbb{R}^d) \) and is such that

\[
\begin{align*}
\int \int f_0(1 + |x|^2 + |v|^2)dxdv &< \infty \\
\int \int f_0 \log f_0 dxdv &< \infty
\end{align*}
\]

Provided these assumptions are satisfied, the following result holds

Theorem 3 (DiPerna and Lions, 1989). Suppose the collision kernel \( q \) and the initial datum \( f_0 \) satisfy Assumptions 1 and 2 respectively. Then there exists a renormalized solution \( f \) of the Boltzmann equation (B) such that \( f \in C(\mathbb{R}^+,L_1^1(\mathbb{R}^d \times \mathbb{R}^d)) \) and the following bounds hold:

\[
\begin{align*}
\int \int f(t,x,v)(1 + |x|^2 + |v|^2)dxdv &\leq \int \int f_0(x,v)(1 + 2|x|^2 + (2t^2 + 1)|v|^2)dxdv \\
\int \int f(t,x,v) \log f(t,x,v)dxdv + \int_0^\infty \int \int e(f)(s,x,v)dxdvds &\leq \int \int f_0(\log f_0 + 2|x|^2 + 2|v|^2) + C_d
\end{align*}
\]

where \( C_d \) is a constant depending only on the dimension \( d \).

We outline below the strategy of the proof.

3.2 Short sketch of the proof

The proof is based on compactness arguments. In particular, a renormalized solution \( f \) will be obtained as the limit of (a subsequence of) solutions \( (f^n) \) to nicer problems \( (B_n) \).

- We start by proving some a priori estimates which follow directly from the structure of (B).
- Define a sequence of truncated problems \( (B_n) \) corresponding to nice initial data \( f_0^n \) and truncated collision operators \( Q^n(f,f) \), and show (using a contraction argument) that they have unique global solutions \( (f^n) \) satisfying the a priori bounds uniformly in \( n \).
- Observe that these bounds imply (by Dunford-Pettis criterion, Theorem 15) that the sequence \( (f^n) \) is weakly compact in \( L_1^1((0,T) \times \mathbb{R}^d \times \mathbb{R}^d) \), so there exists a function \( f \) in the same space such that \( f^n \rightharpoonup f \) up to the extraction of a subsequence. We aim to show that \( f \) is the claimed renormalized solution to (B). To this end, we turn to discuss the convergence of the sequence of truncated collision operators.
- As we are interested in a renormalised solution we turn our attention to \( \frac{Q^n(f^n,f^n)}{1 + f^n} \). We first show that \( \frac{Q^n(f^n,f^n)}{1 + f^n} \) is contained in a weakly compact subset of \( L_1^1((0,T) \times \mathbb{R}^d \times B_R) \) for all \( T,R < \infty \): this is a consequence of DP criterion, the a priori estimates and the careful definition of the collision operators in \( (B_n) \).
• From this, we obtain the compactness of \( \frac{Q^n_+(f^n, f^n)}{1 + f^n} \) via the entropy inequality
\[
Q^n_+(f^n, f^n) \leq KQ^n_+(f^n, f^n) + \frac{1}{\log K} e_n(f^n).
\]

• Hence, for now, \( f^n \in L^1 \) and \( \frac{Q^n_+(f^n, f^n)}{1 + f^n} \) is weakly compact. The problem is that we don’t know whether \( \frac{Q^*_+(f^n, f^n)}{1 + f^n} \) goes to \( Q^+(f, f) \), because of the non linearity.

• On the other hand, with a careful use of the velocity averaging results, we can show that the above convergence holds, even strongly, for the respective velocity averages, i.e.
\[
\int Q^*_+(f^n, f^n)\phi dv \rightarrow \int Q^+(f, f)\phi dv
\]
in \( L^1((0, T) \times \mathbb{R}^d) \), for each compactly supported function \( \phi \in L^\infty((0, T) \times \mathbb{R}^d \times \mathbb{R}^d) \).

• To conclude, we introduce an equivalent form of \( \text{(B)} \), and show that it is satisfied by \( f \). This, together with the right integrability conditions on \( \frac{Q^*_+(f, f)}{1 + f} \), implies that \( f \) is a renormalised solution to \( \text{(B)} \), as claimed.

### 3.3 A priori estimates

Let \( f \) be a solution to \( \text{(B)} \) with initial datum \( f_0 \) satisfying Assumptions \( 2 \). Take \( \psi \in C^\infty(\mathbb{R}^d) \) and assume we have all the necessary decay for the calculations below to make sense.

We start by integrating \( \psi \) against the collision operator:

\[
\int \psi(v)Q(f, f)dv = \int \int q(v - v_*, \omega)\psi(v)(f'_s f' - f_s f)dv_* dv d\omega
\]

(13)

Recall that \( q(v_\ast - v, \omega) = q(v - v_\ast, \omega) \) and exchange \( v \) and \( v_\ast \), to get

\[
\int \psi(v)Q(f, f)dv = \int \int q(v - v_\ast, \omega)\psi(v_\ast)(f'_s f' - f_s f)dv_* dv d\omega
\]

(14)

Now operate in (13) the change of variables \( (v, v_\ast) \mapsto (v, v'_\ast) \), or more explicitly

\[
\begin{align*}
    v' &= v - \omega[\omega \cdot (v - v_\ast)] \\
    v'_\ast &= v_\ast + \omega[\omega \cdot (v - v_\ast)]
\end{align*}
\]

and note that the Jacobian associated with this transformation is 1. We get

\[
\int \psi(v)Q(f, f)dv = \int \int q(v' - v'_\ast, \omega)\psi(v')(f_s f' - f'_s f)dv_* dv d\omega
\]

(15)

where we have used the second relation in (10) for the second equality. Finally we once again exchange \( v \) and \( v_\ast \) in (15) to get

\[
\int \psi(v)Q(f, f)dv = -\int \int q(v - v_\ast, \omega)\psi(v'_\ast)(f'_s f' - f_s f)dv_* dv d\omega
\]

(16)

Adding up equations (13) - (16) we obtain the identity:

\[
\int \psi(v)Q(f, f)dv = \frac{1}{4} \int \int q(v - v_\ast, \omega)(f'_s f' - f_s f)[\psi + \psi_\ast - \psi' - \psi'_\ast]dv_* dv d\omega
\]

(17)

This motivates the following definition.
Definition 17. A function $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is said to be collision invariant if
\[ \phi + \phi = \phi' + \phi' \ast. \]

This definition has a simple interpretation. Observe that, by (17), if $\psi$ is collision invariant then
\[ \int Q(f, f) \phi(v) dv = 0. \]

Since this integral represents the rate of change in the average value of $\phi$ due to collisions, the above relation tells us that this average value stays unchanged.

Then (8) immediately implies that any function of the form
\[ \psi(v) = a + b \cdot v + c|v|^2 \]  

is collision invariant and hence gives 0 when integrated against the collision operator. To make use of that, we go back to equation (B), multiply it by (18) and integrate with respect to $v$ to get
\[ 0 = \int Q(f, f) \psi(v) dv = \int \psi \left( \frac{\partial f}{\partial t} + v \cdot \nabla_x f \right) dv \]
\[ = \frac{\partial}{\partial t} \int \psi f dv + \text{div}_x \left( \int \psi v f dv \right) \]

Next, we integrate over $x \in \mathbb{R}^d$ and use the total time-derivative to see that:
\[ \frac{d}{dt} \int \psi f dv = 0. \]

This gives the following conservation laws:

- **Conservation of mass**
\[ \int \int f(t, x, v) dx dv = \int \int f_0(x, v) dx dv \]  

- **Conservation of momentum**
\[ \int \int f(t, x, v) v dx dv = \int \int f_0(x, v) v dx dv \]  

- **Conservation of kinetic energy**
\[ \int \int f(t, x, v)|v|^2 dx dv = \int \int f_0(x, v)|v|^2 dx dv \]  

Another conservation law is obtained integrating the Boltzmann equation against $\psi(t, x, v) = |x - tv|^2$ with respect to $x$ and $v$. By (8) and (10), in fact, we deduce that
\[ |x - tv|^2 + |x - tv'|^2 = 2|x|^2 + t^2(|v|^2 + |v'|^2) - 2t x \cdot (v' + v) \]
\[ = 2|x|^2 + t^2(|v|^2 + |v|^2) - 2t x \cdot (v + v') \]
\[ = |x - tv|^2 + |x - tv|^2 \]
so that $\int Q(f, f)|x - tv|^2 dv = 0$. Apply the same reasoning as before to get
\[ \frac{d}{dt} \int \int f(t, x, v)|x - tv|^2 dx dv = 0 \]
and hence the conservation law
\[ \int \int f(t, x, v)|x - tv|^2 dx dv = \int \int f_0(x, v)|x|^2 dx dv. \]

\[ ^1 \text{In fact, it can be shown that all collision invariants have this form, see \cite{2} for details.} \]
3.3.1 Entropy identity

Multiply the Boltzmann equation by \((1 + \log f)\) and integrate over \(v\):

\[
\int (1 + \log f)(\partial_t f + v \cdot \nabla_x f)dv = \int Q(f, f)(1 + \log f)dv
\]

So

\[
\int Tfdv + \partial_t \left[ \int f \log f dv \right] + \text{div}_x \left[ \int v \cdot f \log f\right] = \int Q(f, f)dv + \int Q(f, f)\log f dv
\]

and using that \(f\) solves the Boltzmann equation we get

\[
\partial_t \left[ \int f \log f dv \right] + \text{div}_x \left[ \int v \cdot f \log f\right] = \int Q(f, f)\log f dv
\]

\[
= -\frac{1}{4} \int \int q(v - v_\ast, \omega)(f^\prime f^\prime - f_\ast f_\ast) \log \frac{f f^\prime}{f_\ast f_\ast} dv dv \omega.
\]

where in the last equality we have used relation \([17]\) with \(\psi = \log f\). Now integrate over \(x \in \mathbb{R}^d\), to obtain the identity

\[
\frac{d}{dt} \int \int f \log f dv dv + \int \int e(f)dv dv = 0 \tag{23}
\]

with

\[
e(f) = \frac{1}{4} \int \int q(v - v_\ast, \omega)(f^\prime f^\prime - f_\ast f_\ast) \log \frac{f f^\prime}{f_\ast f_\ast} dv dv \omega. \tag{24}
\]

Note that the non-negativity of \(e(f)\) implies that \(\int \int f \log f\) is non-increasing in time.

**Lemma 8.** Let \(g\) be any non negative function on \(\mathbb{R}^d \times \mathbb{R}^d\) such that \(\int \int g|\log g|dx dv < \infty\). Then

\[
\int \int g|\log g|dx dv \leq \int \int g \log g dx dv + 2 \int \int g(|x|^2 + |v|^2)dx dv + C_d
\]

where \(C_d\) is a constant depending only on the dimension \(d\) of the space.

**Proof.** Start by splitting the integral

\[
\int \int g|\log g|dx dv = \int \int g \log g dx dv - 2 \int \int g \log g dx dv \tag{25}
\]

Then notice that

\[
g \log g \mathbf{1}_{(g \in (0,1))} = g \log g \mathbf{1}_{\{e^{-|x|^2 + |v|^2} \leq g \leq 1\}} + g \log g \mathbf{1}_{\{g < e^{-|x|^2 + |v|^2}\}}
\]

\[
ge = g \log g \mathbf{1}_{\{e^{-|x|^2 + |v|^2} \leq \log g < 0\}} + g \log g \mathbf{1}_{\{\log g > |x|^2 + |v|^2\}}
\]

\[
\geq -g(|x|^2 + |v|^2) \mathbf{1}_{\{|x|^2 + |v|^2 \leq \log g < 0\}} - g \log g \frac{1}{g} \mathbf{1}_{\{g < e^{-|x|^2 + |v|^2} \}}
\]

where \(\mathbf{1}_{(\cdot)}\) denotes the indicator function. By substituting this into \([25]\) we get

\[
\int \int g \log g dx dv = \int \int g \log g dx dv - 2 \int \int g \log g dx dv
\]

\[
\leq \int \int g \log g dx dv + 2 \int \int g(|x|^2 + |v|^2)dx dv + 2 \int \int g \log g \frac{1}{g} \mathbf{1}_{\{g < e^{-|x|^2 + |v|^2} \}} dx dv.
\]

By the inequality \(t \log \frac{1}{t} \leq C \sqrt{t}\), which holds for \(t \in (0, 1)\), we can bound the last term from above so that

\[
\int \int g \log g \frac{1}{g} \mathbf{1}_{\{g < e^{-|x|^2 + |v|^2} \}} dx dv \leq C \int \int e^{-\frac{1}{2}(|x|^2 + |v|^2)} dx dv =: C_d
\]

which completes the proof of the lemma. \(\square\)
Apply the above result with $g = f(t, x + vt, v) = f^2(t, x, v)$, to get
\[
\int \int f^2 |\log f|^3 dx dv \leq \int \int f^4 \log f^2 dx dv + 2 \int \int f^4 (|x|^2 + |v|^2) dx dv + C_d
\]
and the change of variables $(x, v) \mapsto (x - vt, v)$ gives us
\[
\int \int f |\log f|^3 dx dv \leq \int \int f \log f dx dv + 2 \int \int f(|x - vt|^2 + |v|^2) dx dv + C_d.
\] (26)

**Lemma 9. (A priori estimates)**

Suppose that $q$ is a non negative measurable function in $L^\infty_{loc}(\mathbb{R}^d \times \mathbb{R}^d \times S^{d-1})$, $q = q(x, |v - v_\ast|, |(v - v_\ast) \omega|)$ and $q$ grows at most polynomially in $x$ and $v - v_\ast$. Assume further that $f$ is a positive solution to the Cauchy problem (3) with initial datum satisfying Assumptions [2] such that $f \in C^1((0, \infty), \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d))$, where $\mathcal{S}$ denotes the Schwartz space. If, moreover, $|\log f|$ grows at most polynomially in $(x, v)$ uniformly in compact time intervals, then the following hold:
\[
\int \int f(t, x, v) dx dv = \int \int f_0(x, v) dx dv
\]
\[
\int \int f(t, x, v) |v|^2 dx dv = \int \int f_0(x, v) |v|^2 dx dv
\]
\[
\int \int f(t, x, v) |x - tv|^2 dx dv = \int \int f_0(x, v) |x|^2 dx dv
\]
\[
\int \int f(t, x, v) \log f(t, x, v) dx dv + \int_0^t \int \int e(s) f dx dv ds = \int \int f_0 \log f_0 dx dv.
\]

These identities in turn imply:
\[
\int \int f(t, x, v) (1 + |x|^2 + |v|^2) dx dv \leq \int \int f_0(x, v) (1 + 2|x|^2 + (2t^2 + 1)|v|^2) dx dv
\] (27)
\[
\int \int f(t, x, v) \log f(t, x, v) dx dv + \int_0^\infty \int \int e(f)(s, x, v) dx dv ds \leq \int \int f_0(|\log f_0| + 2|x|^2 + 2|v|^2) + C_d
\] (28)

Note that the assumptions of the lemma are enough to carry out all the a priori computations made in the previous section, and these computations are not affected by allowing the collision kernel to depend on the position. The assumptions on the growth of the logarithm is needed to make sure that integrals such as $\int \int f \log f$ are convergent, as $f$ being in $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ in $(x, v)$ potentially allows for a decay that is too rapid for the integral to converge.

Let us now turn to the proof.

**Proof.** The first three relations have already been shown. The fourth is obtained integrating (23) in the time interval $[0, t]$. It only remains to prove (27) and (28).

For the first one we split the integral, and use the conservation of mass and kinetic energy to get
\[
\int \int f(t, x, v) (1 + |v|^2) dx dv = \int \int f_0(x, v) (1 + |v|^2) dx dv
\]
To bound the remaining term we use the inequality $|x|^2 \leq (|x - tv| + t|v|)^2 \leq 2(|x - tv|^2 + t^2|v|^2)$ and (21)-(22), to get
\[
\int \int f(t, x, v) |x|^2 dx dv \leq 2 \int \int f_0(x, v) (|x|^2 + t^2|v|^2) dx dv
\]
which yields the required result. Finally, for the last inequality combine (26) with (23) and use the conservation laws to get
\[
\int \int f(t, x, v) \log f(t, x, v) dx dv \leq \int \int f \log f dx dv + 2 \int \int f(|x - vt|^2 + |v|^2) dx dv + C_d
\]
\[
= - \int_0^t \int \int e(f)(s) dx dv ds + \int \int f_0 \log f_0 dx dv + 2 \int \int f_0(|x|^2 + |v|^2) dx dv + C_d
\]
\[
\leq - \int_0^\infty \int \int e(f)(s) dx dv ds + \int \int f_0 \log f_0 dx dv + 2 \int \int f_0(|x|^2 + |v|^2) dx dv + C_d
\]
where in the last inequality we have used the fact that \( \int t f \log f \) is non-increasing in time, and the lemma is proved.

### 3.4 Truncated problems

We now want to make use of the a priori estimates. The idea is to define some *nicer* problems, for which we can prove existence and uniqueness of solutions in the class of functions satisfying all the assumptions of Lemma 3. For these solutions the conclusions of the above lemma will still hold.

To implement this strategy, we have to identify the problems in proving the existence of a solution to \((B)\), and find a way to avoid them. These are the following:

- Large relative velocity collisions.
  - Note that Assumptions 1 on the collision kernel do not control its behaviour for large velocities.
- Non-linearity of the collision operator.
  - In order to deal with the convergence of \(Q(f,f)\) we will introduce a suitable additional renormalization, aimed to slow down the growth of the collision operator.
- Regularity of the initial datum.
  - Lemma 9 takes the initial datum in the Schwartz space, but the theorem claims the existence of a solution when \(f_0\) is merely \(L^1_{\infty}(\mathbb{R}^d \times \mathbb{R}^d)\). We will have to use some approximation argument.

We take care of these problems as follows:

Assume that the collision kernel \(q\) satisfies Assumptions 1, and consider a sequence of truncated kernels \(q_n\) such that:

- \(q_n \in C^\infty_{0,\infty}(\mathbb{R}^d \times S^{d-1})\) for all \(n \geq 1\),
- the \((q_n)\) satisfy the second and third assumptions in (1) uniformly in \(n\),
- each \(q_n\) vanishes for \(|(v - v_\ast) \cdot \omega| < \delta_n\), where \((\delta_n)\) is a sequence of positive real numbers such that \(\delta_n \searrow 0\),
- \(q_n \longrightarrow q\ a.e.\)

For each \(n\) we then define the truncated (and renormalized) collision operator

\[
Q^n(f,f) = \frac{\int \int q_n(f'f'_\ast - ff_\ast)d\omega dv}{1 + \delta_n \int |f|dv}.
\]

Finally, if \(f_0\) satisfies Assumptions 2 we approximate it by a sequence \((f^n_0)\) such that:

- \(f^n_0 \in S(\mathbb{R}^d \times \mathbb{R}^d)\) for all \(n \geq 1\),
- for each \(f_n\), there exists a positive real number \(\mu_n\) such that \(f^n_0 \geq \mu_ne^{-|x|^2 + |v|^2}\),
- \[
\int \int f^n_0(1 + |x|^2 + |v|^2)dxdv \longrightarrow \int \int f_0(1 + |x|^2 + |v|^2)dxdv \quad \text{as } n \to \infty \quad (29)
\]
- \[
\int \int f_0|\log f_0|dxdv \longrightarrow \int \int f_0|\log f_0|dxdv \quad \text{as } n \to \infty \quad (30)
\]

We have then obtained the sequence of truncated problems

\[
\begin{cases}
\frac{\partial f^n}{\partial t} + v \cdot \nabla_x f^n = Q^n(f^n,f^n) \\
f^n(0,\cdot) = f^n_0
\end{cases}
\]

(Bn)

Recall that \(A(z) = \int_{S^{d-1}} q(z,\omega)d\omega\) and set

\[
A_n(z) = \int_{S^{d-1}} q_n(z,\omega)d\omega
\]
Applying Tonelli’s theorem yields
\[ Q^n(f^n, f^n) = Q^n(f^n, f^n) - Q^n(f^n, f^n) \]
where \( Q^n(f^n, f^n) = \frac{f^n(A_n \ast f^n)}{1 + \delta_n f^n dv} \).

**Proposition 4.** For each \( n \geq 1 \), the Cauchy problem \([Bn]\) has a unique global solution \( f^n \in C^1((0, \infty), S(\mathbb{R}^d \times \mathbb{R}^d)) \). Moreover, \( f^n \) satisfies all the assumptions of Lemma 9 and hence the conservation laws \([19], [22]\) and inequalities \([27], [28]\).

To prove this result, we will establish the following lemma which we will use to bound \( \int Q^n(f_f) dv \).

**Lemma 10.** Suppose \( a, b \in L^1(\mathbb{R}^d) \). Then
\[
\int \int \int q_n |a| |b| d\omega dv dv_1 = \int \int \int q_n |a(v_1 - \omega \cdot (v_1 - v_\ast))| |b(v_\ast + \omega \cdot (v_1 - v_\ast))| d\omega dv dv_1 
\leq c_n ||a||_{L^1} ||b||_{L^1}
\]
for some constant \( c_n \).

**Proof.** To prove this result, we simply make the change of variables \( u(\omega) = \omega \cdot (v_1 - v_\ast) \) for fixed \( v_1 \) and \( v_\ast \). The Jacobian associated with this transformation is given by \( (\omega \cdot (v_1 - v_\ast))^d \) and so
\[
\int \int \int q_n |a| |b| d\omega dv dv dv_1 = \int \int \int q_n |a(v_1 - u)| |b(v_\ast + u)| \frac{1}{|\omega \cdot (v_1 - v_\ast)|^d} d\omega dv dv_1
\]
Note that the integrand is defined everywhere, as \( q_n \) vanishes if \( |\omega \cdot (v_1 - v_\ast)| < \delta_n \). Thus
\[
\int \int \int q_n |a| |b| d\omega dv dv dv_1 \leq \frac{1}{\delta_n} \int \int \int q_n |a(v_1 - u)| |b(v_\ast + u)| d\omega dv dv_1
\]
Moreover, \( q_n \) is supported only on a compact set of \( u \), which we will denote by \( K_n \). Additionally, \( q_n \) is bounded on this set by some constant \( \alpha_n \). Thus
\[
\int \int \int q_n |a| |b| d\omega dv dv dv_1 \leq \frac{\alpha_n}{\delta_n} \int \int \int_{K_n} |a(v_1 - u)| |b(v_\ast + u)| d\omega dv dv_1
\]
Applying Tonelli’s theorem yields
\[
\int \int \int q_n |a| |b| d\omega dv dv dv_1 \leq \frac{\alpha_n}{\delta_n} \int \int \int_{K_n} |a(v_1 - u)| |b(v_\ast + u)| dv_1 du 
\leq \frac{\alpha_n}{\delta_n} \int_{K_n} \left( \int |a(v_1 - u)| dv_1 \right) \left( \int |b(v_\ast + u)| dv_\ast \right) du 
\leq \frac{\alpha_n}{\delta_n} ||a||_{L^1} ||b||_{L^1} \lambda(K_n)
\]
and setting \( c_n = \frac{\alpha_n}{\delta_n} \lambda(K_n) \) yields the desired inequality.

We will use this lemma to establish \( L^1 \) bounds on the truncated collision kernel.

**Lemma 11.** There are constants \( C_n \) such that for any \( f, g \in L^1 \cap L^\infty(\mathbb{R}^d) \), we have
\[
\int |Q^n(f, f)(v) - Q^n(g, g)(v)| dv \leq C_n ||f - g||_{L^1}
\]
and
\[
\int |Q^n(f, f)(v)| dv \leq C_n ||f - g||_{L^1}
\]
Proof. To show this, we split the collision kernel into $Q^+_n$ and $Q^-_n$ where

$$Q^+_n(f, f) = \frac{\int \int q_n(f' f^*) \, d\omega dv}{1 + \delta_n \int |f| \, dv}.$$  

and

$$Q^-_n(f, f) = \frac{\int \int q_n(f f^*) \, d\omega dv}{1 + \delta_n \int |f| \, dv}.$$  

It is simple to see that

$$Q^+_n(f, f) - Q^-_n(g, g) = \frac{\int \int \int \left(1 + \delta_n \int |g| \, dv \right) (f' f^*) - (1 + \delta_n \int |f| \, dv) (g' g^*) \, d\omega dv}{1 + \delta_n \int |f| \, dv \, (1 + \delta_n \int |g| \, dv)}$$

We will bound the term

$$\left| \delta_n(f' f^*) \int |g| \, dv - g' g^* \int |f| \, dv \right|$$

To do this, we first perform the following simple calculation

$$\left| \delta_n(f' f^*) \int |g| \, dv - g' g^* \int |f| \, dv \right| = \left| \delta_n \int f' f^* |g| - g' g^* |f| \, dv \right|$$

$$\leq \delta_n \left( \int f' \left( |g| - |f| \right) \, dv + \int f' g^* (|g| - |f|) \, dv + \int f' g^* (f' - g) \, dv \right)$$

Each of our terms involves exactly one of $f', g'$ or $(f' - g)$, exactly one of $f^*, g^*$ or $(f^* - g^*)$ and exactly one of $|f|_{L^2}, |g|_{L^2}$ or $||f' - g'||_{L^2}$. This was done deliberately so that we can apply Lemma 10 to see that

$$\int \int q_n \left( f' f^* \int |g| \, dv - g' g^* \int |f| \, dv \right) \, d\omega dv, dv_1 \leq 3\epsilon_n \delta_n \left( \int f \, dv \right)$$

Similar arguments yield

$$\int \int q_n (f' f^* - g' g^*) \, d\omega dv, dv_1 \leq \beta_n \left( \int (f \, dv) \right)$$

and so

$$\int |Q^+_n(f, f)(v) - Q^-_n(g, g)(v)| \, dv \leq \frac{||f - g||_{L^2} \left( \beta_n (||f||_{L^2} + ||g||_{L^2}) + 3\epsilon_n \delta_n \left( \int ||f||_{L^2} \, dv \right) \right)}{1 + \delta_n (||f||_{L^2} + ||g||_{L^2}) + \delta_n (||f||_{L^2})}$$

$$\leq C_n ||f - g||_{L^2}$$

A slightly easier argument (there is no need to change coordinates) can be used to show that

$$\int |Q^+_n(f, f)(v) - Q^-_n(g, g)(v)| \, dv \leq D_n ||f - g||_{L^2}$$

Combining these two results leaves us

$$\int |Q^+(f, f)(v) - Q^-(g, g)(v)| \, dv \leq C_n ||f - g||_{L^2}$$

The second bound follows by taking $g = 0$. 

We can now prove Proposition 4.
Proof. We construct the following scheme to approximate the required $f^n$. We define the sequence $(f_m)$ recursively by

\[
T f_0 = 0 \\
f_0(0,x,v) = f_0^0(x,v) \\
T f_{m+1} = Q^n(f_m,f_m) \\
f_{m+1}(0,x,v) = f_{m+1}^0(x,v)
\]

We can explicitly compute these as

\[
f_0^0(t,x,v) = f_0^0(x,v) \\
f_m^0(t,x,v) = f_0^0(x,v) + \int_0^t Q^n(f_m^0,f_m^0) \, ds
\]

We will show that the operator that maps $f_m \mapsto f_{m+1}$ is a contraction on $C([0,\tau]; L^1_{x,v})$ for sufficiently small $\tau$. Then we have a solution to the integral form of the equation. In other words a mild solution.

To prove that $f_m$ is non-negative for some time interval, observe that $f_m \in C[0,T]$ has initial condition $f_m(0,x,v) = f_0^0(x,v) > 0$. Thus $f_m$ is non-negative for some time interval. Hence (on this time interval)

\[
T f_{m+1} = Q^n(f_m,f_m) \\
= Q^n(f_m,f_m) - Q^n(f_m,f_m) \\
\geq -m(A_n \ast f_m) \\
\frac{1}{1 + b_n m f_m} \\
\geq C_n f_m
\]

Now because $\frac{(A_n \ast f_m)}{1 + b_n f_m} \geq C_n$ for some $C_n > 0$, we have

\[
T f_{m+1} \geq -C_n f_m
\]

So we may take $\tau$ small enough for this to be a contraction. Then by the fixed point theorem we have a solution in $[0,\tau]$. As none of our estimates depended on the norm of the initial condition we may then restart the process at $\tau/2$ say, and continue to get existence for all time.

In fact we can prove that the limit is in $C([0,\tau], S(\mathbb{R}^d \times \mathbb{R}^d))$. To do this we bound the $x$ and $v$ derivatives of $f_{m+1}$ by moving them under the integral. When the derivatives hit $q_n$ they produce a function that obeys all the same bounds as $q_n$, and when they hit $f_m$ they produce its derivatives. In this way the same argument gives that any derivative of the solution is also in $C([0,\tau], S(\mathbb{R}^d \times \mathbb{R}^d))$. This argument also works for multiplication by polynomials: we see that a polynomial in $v$ can be absorbed into the collision kernel which still obeys the same assumptions as before, and a polynomial in $x$ is taken care of by the fact that the initial data is Schwartz. This is enough to give the Schwartz space result.

Finally we argue that as the limit obeys $f_{m+1}(t,x,v) = f_0^0 + \int_0^t Q^n(f_m,f_m) \, ds$, and the integrand is smooth and compactly supported we can deduce that $f$ is time differentiable, i.e. $f \in C^1([0,\tau], S(\mathbb{R}^d \times \mathbb{R}^d))$. 

To prove that $|\log f_m|$ grows at most polynomially in $(x,v)$, observe that $f_m \in C[0,T]$ has initial condition $f_m(0,x,v) = f_0^0(x,v) > 0$. Thus $f_m$ is non-negative for some time interval. Hence (on this time interval)

\[
T f_{m+1} = Q^n(f_m,f_m) \\
= Q^n(f_m,f_m) - Q^n(f_m,f_m) \\
\geq -m(A_n \ast f_m) \\
\frac{1}{1 + b_n m f_m} \\
\geq C_n f_m
\]

Now because $\frac{(A_n \ast f_m)}{1 + b_n f_m} \geq C_n$ for some $C_n > 0$, we have

\[
T f_{m+1} \geq -C_n f_m
\]
Send $m \to \infty$ to get the inequality
\[ T^n f \geq C_n f. \]
This implies $\partial_t (f^n) = (T^n f)^{\pm} \geq C_n f^{\pm}$ and hence, by Gronwall’s inequality, we obtain
\[ f^{\pm} \geq f^0(t, x, v) e^{-C_n t}. \]
Going back along the characteristic, we finally get
\[ f^n \geq f^0(t - v t, v) e^{-C_n t} \]
as claimed.  \hfill \Box

### 3.5 Weak compactness results

For now, we have a sequence $(f^n)$ of solutions to the problems $\{B_n\}$ for which all the a priori estimates, together with relations (29)-(30), hold. In this section we want to show that the two sequences
\[ f^n, \quad \frac{Q_n^\pm(f^n, f^n)}{1 + f^n} \quad n \geq 1 \]
are contained in weakly compact sets: this will allow to apply the velocity averaging results in a later step.

The main tool we will use to prove weak compactness is the Dunford-Pettis (DP) criterion (Theorem 15). In particular, we will make use of it in the following form:

**Corollary 3.** Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function, $w : U \to \mathbb{R}_+$ be in $L^\infty_{\text{loc}}(U)$, and assume $h(s)/s \to \infty$ as $s \to \infty$ and $w(y) \to \infty$ as $|y| \to \infty$. If, then, $(f^n)$ is a sequence which is uniformly bounded in $L^1(U)$ and such that
\[ \sup_n \int_{\mathbb{R}^d} [h(|f^n|) + |f^n|(1 + w)] dy < \infty \]
then $(f^n)$ is uniformly integrable, and hence relatively weakly compact in $L^1(U)$.

#### 3.5.1 Weak compactness of $(f^n)$

The weak compactness of the sequence $(f^n)$ of solutions to the truncated problems follows immediately from the a priori estimates.

**Lemma 12.** The sequence $(f^n)$ of solutions to $\{B_n\}$ is weakly compact in $L^1((0, T) \times \mathbb{R}^d \times \mathbb{R}^d)$.

**Proof.** Observe that combining estimates (27)-(28) with equations (29)-(30) we have
\[ \sup_n \sup_{t \in (0, T)} \int \int f^n(1 + |x|^2 + |v|^2 + |\log f^n|) dxdv \leq C_T \]
for all $T > 0$. The claim then follows by applying Corollary 3 with $U = (0, T) \times \mathbb{R}^d \times \mathbb{R}^d$ and
\[ h(f^n) = f^n|\log f^n| \]
\[ w(x, v) = |x|^2 + |v|^2. \]

Up to the extraction of a subsequence (that we still call $(f^n)$), hence
\[ f^n \to f \quad \text{in} \quad L^1((0, T) \times \mathbb{R}^d \times \mathbb{R}^d). \]
3.5.2 Weak compactness of the collision term

**Theorem 5.** For any $T > 0$, $R > 0$, the sequences

$$
\frac{Q^n(f^n, f^n)}{1 + f^n} \quad \frac{Q^n(f^n, f^n)}{1 + f^n} \quad n \geq 1
$$

are relatively weakly compact in $L^1((0,T) \times \mathbb{R}^d \times B_R)$, where $B_R = \{ v \in \mathbb{R}^d : \|v\| \leq R \}$.

To prove this, we start from the negative part $Q^n(f^n, f^n)$, and show that it is a uniformly integrable sequence, so that weak compactness follows from DP criterion. We then obtain control on the positive part via the following inequality.

**Lemma 13** (Entropy inequality). For all $K \geq 1$ we have

$$
Q^n(f^n, f^n) \leq KQ^n(f^n, f^n) + \frac{1}{\log K} c_n(f^n) \quad \forall K \geq 1,
$$

where $c_n$ is defined as in replacing $q$ with $q_n$.

Note that this remarkable inequality gives us control on the global part of the collision operator in terms of the local one, and vice versa. This is, as will become clear from the proof, due to the particular form of the collision kernel, and is therefore characteristic of the Boltzmann equation.

**Proof.** Since $n$ is fixed, we drop it in the notation. Define the events

$$
A_K = \{ (v, v_*, \omega) : f' f'_* \geq K f f_* \}
$$

$$
B_K = A_K^c.
$$

Then

$$
Q_+(f, f) = \int \int q(v - v_*, \omega)f' f'_* [1_{A_K} + 1_{B_K}] dv_* d\omega
$$

$$
\leq K \int \int q(f f_*) 1_{B_K} dv_* d\omega + \int \int q(f' f'_* - f f_*) 1_{A_K} dv_* d\omega + \int \int q(f f_*) 1_{A_K} dv_* d\omega
$$

$$
\leq K \int \int q(f f_*) 1_{B_K} dv_* d\omega + \frac{1}{\log K} \int \int q(f' f'_* - f f_*) \log \frac{f' f'_*}{f f_*} 1_{A_K} dv_* d\omega + K \int \int q(f f_*) 1_{A_K} dv_* d\omega
$$

$$
= KQ_-(f, f) + \frac{1}{\log K} \int \int q(f' f'_* - f f_*) \log \frac{f' f'_*}{f f_*} 1_{A_K} dv_* d\omega
$$

as claimed. \hfill \Box

We are now ready to prove Theorem 5.

**Proof.** Let us start by proving that the sequence $\frac{Q^n(f^n, f^n)}{1 + f^n}$ satisfies the DP criterion in $L^1((0,T) \times \mathbb{R}^d \times B_R)$, and is therefore weakly compact. Note that, since

$$
\frac{Q^n(f^n, f^n)}{1 + f^n} \leq \frac{f^n}{1 + f^n} \cdot \frac{A_n * f^n}{1 + \delta_n f^n d} \leq A_n * f^n
$$

it suffices to prove that the sequence $A_n * f^n$ is uniformly integrable, i.e. it satisfies properties (a)-(b)-(c) of Definition 18.

To show (a) observe that

$$
\int_{\mathbb{R}^d} \int_{B_R} (A_n * f^n) dv dx = \int_{\mathbb{R}^d} \int_{B_R} \int_{\mathbb{R}^d} A_n(v - z) f^n(z) dz dv dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f^n(x, z) \left( \int_{B_R} A_n(z - v) dv \right) dz dx
$$

$$
\leq C \int_{\mathbb{R}^d} \int_{B_R} f^n(x, z)(1 + |z|^2) dz dx = C \int_{\mathbb{R}^d} \int_{B_R} f^n(x, z)(1 + |z|^2) dz dx
$$

25
by \[12\], and the last term is uniformly bounded in \(n\) by \[29\]. Let us now move to conditions (b) and (c). We split the proof in several steps.

**Step 1**
Assume \(\|A\|_{L^1(\mathbb{R}^d)} < \infty\). Then
\[
\|A\|_{L^1(\mathbb{R}^d)} = \|q_n\|_{L^1(\mathbb{R}^d \times S^{d-1})} = a_n
\]
and, using the properties of the \(q_n\), we can assume the \(a_n\) to be uniformly bounded in \(n\). Let us now introduce the function \(\phi(s) = s \log s \wedge 0\), and observe that to show (b) it is enough to prove that \(\phi(f^n) \in L^\infty((0,T); L^1(\mathbb{R}^d \times B_R))\). To this end note that \(\phi(s) \leq s \log s = s (\log \frac{s}{2} + \log a) = a \phi \left(\frac{s}{2}\right) + s |\log a| + \log a < a \phi \left(\frac{s}{2}\right) + s |\log a|\) for all \(a \geq 0\). Apply this inequality with \(a = a_n\) yields
\[
\int_{\mathbb{R}^d} \int_{B_R} \phi(A_n * f^n) dv dx \leq a_n \int_{\mathbb{R}^d} \int_{B_R} \frac{A_n * f^n}{a_n} dv dx + |\log a_n| \int_{\mathbb{R}^d} \int_{B_R} A_n * f^n dv dx
\]
\[
\leq \int_{\mathbb{R}^d} \int_{B_R} A_n * \phi(f^n) dv dx + |\log a_n| a_n \|f^n\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)}
\]
\[
\leq a_n \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(f^n) dv dx + |\log a_n| a_n \|f^n\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)}
\]
The last term is then uniformly bounded in \(n\), since
- \(a_n, |\log a_n|\) are uniformly bounded
- \(\int \phi(f^n) dv dx \leq f^n|\log f^n| dv dx \leq \int \frac{f^n}{a_n} \left(|\log f^n| + 2|x|^2 + 2|v|^2\right) + C d\) by \[28\], and the uniform bound for the RHS follows from \[29\]–\[30\].
- \(\|f^n\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} = \|f^n\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)}\) and the sequence \(\|f^n\|_{L^1}\) is uniformly bounded because of \[29\].

In conclusion, since this holds for each fixed \(t \in (0,T)\), we have proved that \(\phi(f^n)\) is uniformly bounded in \(L^\infty((0,T); L^1(\mathbb{R}^d \times B_R))\), and (b) follows\(^2\).

We now move to (c). Observe that it is enough to show that
\[
\int_{\mathbb{R}^d} \int_{|v| > K} (A_n * f^n) dv dx \longrightarrow 0 \quad \text{as } K \to \infty
\]
uniformly in \(n\) (and \(t\)). To this end, split the integral as follows:
\[
\int_{\mathbb{R}^d} \int_{|v| > K} (A_n * f^n) dv dx \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} A_n(v - v_*) f^n(x,v_*) 1_{|v_*| \geq K/2} dv_* dx
\]
\[
+ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_{|v_*| \leq K/2} A_n(v - v_*) f^n(x,v_*) dv_* dx
\]
\[
= I + II
\]
For the first term, integrating over \(v\) and using the a priori bound \[21\], we get
\[
I \leq a_n \int_{\mathbb{R}^d} \int_{|v| \geq K/2} f^n(x,v_*) 1_{|v_*| \geq K/2} dv_* dx \leq \frac{a_n}{(K/2)^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |v_*|^2 f^n(x,v_*) dv_* dx
\]
\[
= \frac{4a_n}{K^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |v_*|^2 f^n(x,v_*) dv_* dx \leq \frac{C}{K^2} \to 0 \quad \text{as } K \to \infty
\]
where, by \[29\], the constant \(C\) is independent of \(n\). For the second one:
\[
II \leq \left(\int_{|z| \geq K/2} A_n(z) dz\right) \|f^n\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} = \left(\int_{|z| \geq K/2} A_n(z) dz\right) \|f^n_0\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)}
\]
\[
\leq C' \left(\int_{|z| \geq K/2} A_n(z) dz\right)
\]
\(^2\)Note that, as pointed out in \[2\], we could have set \(R = \infty\) in this argument. We will need, though, \(R\) to be finite in the next step, where we remove the integrability assumption on \(A\).
where $C'$ is a uniform bound on $\|f^n\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)}$ again given by (29). From the integrability assumption on $A$, then, we deduce that the last term then becomes arbitrarily small as $K \to \infty$, and (e) follows.

**Step 2**
We now remove the integrability condition on $A$. Note that if the $A_n$’s are all supported in a compact set then all the previous calculations continue to hold, and we are done. If not, take $K > 0$ and truncate the $A_n$’s by defining

$$A_{n,K}(\varepsilon) := A_n(\varepsilon) \cdot 1_{(|\varepsilon| \leq K)}.$$

Then weak compactness will follow from

$$\sup_n \|A_{n,K} * f^n - A_n * f^n\|_{L^\infty((0,T);L^1(\mathbb{R}^d \times B_R))} \to 0 \quad \text{as } K \to \infty \quad (32)$$

To see that, we calculate explicitly that

$$\|A_{n,K} * f^n - A_n * f^n\|_{L^1(\mathbb{R}^d \times B_R)} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} A_n(v - v_*) 1_{|v - v_*| \geq K} f^n(x, v_*) dv_* dx$$

and observe that if $K > R$ then

$$|v_*| = |v_* - v + v| \geq |v_* - v| - |v| \geq K - R$$

so that

$$1_{(|v_* - v| \geq K)} \leq 1_{(|v_*| \geq K - R)}.$$

Using this inequality and integrating then $A_n$ with respect to $v$ we get

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} A_n(v - v_*) dv_* dx \right| \leq \frac{\varepsilon}{C} \frac{\delta}{K - R} f^n(x, v_*) dv_* dx,$$

and (32) follows by letting $K \to \infty$ and then $\varepsilon \to 0$, and observing that the convergence is uniform in $n$, again by (29).

**Step 3** We deduce the weak compactness of the (renormalized) positive part of the collision operator from that of the negative part via entropy inequality, by noting that for each $R > 0$ and $A$ Borel set of Lebesgue measure $\lambda(A)$, (31) implies

$$\sup_n \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{Q_n(f^n, f^n)}{1 + f^n} \left[ 1_A + 1_{(|\varepsilon| + |v_*| \geq R)} \right] dv_* dx dt \to 0 \quad \text{as } \lambda(A) \to 0, \ R \to \infty.$$

and uniform integrability follows.

\[ \square \]

### 3.5.3 Temporal regularity of $f$

Combining the weak compactness of the above sequences with the a priori estimates, we can obtain some information about the regularity in time of the limit $f$. Our goal is to show that

$$f \in C([0,T), L^1(\mathbb{R}^d \times \mathbb{R}^d)).$$

We start by introducing, for any fixed $\delta > 0$, the new sequence

$$g^n_\delta = \frac{1}{\delta} \log(1 + \delta f^n) \quad (33)$$

and observing that each $g^n_\delta$ solves

$$T g^n_\delta = \frac{1}{1 + \delta f^n} Q^n(f^n, f^n),$$

where $T$ denotes, as usual, the transport operator.
Lemma 14. If the sequence \( (g^n_\delta) \) is defined as in (33), it holds:

\[
\sup_n \sup_{t \in (0,T)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |g^n_\delta - f^n| \, dx \, dv \longrightarrow 0 \quad \text{as } \delta \searrow 0 \tag{34}
\]

Proof. Consider \( \beta_\delta(s) = \frac{s}{1+\delta s} \), so that \( g^n_\delta = \beta_\delta(f^n) \). Note that \( \frac{d}{dt} \beta_\delta(s) = \frac{1}{1+\delta s} \), so that for each \( s > 0 \), \( \beta_\delta(s) \geq \frac{s}{1+\delta s} \). For any \( R > 0 \), therefore, we have

\[
(s - \beta_\delta(s)) = (s - \beta_\delta(s)) \mathbf{1}_{(s < R)} + (s - \beta_\delta(s)) \mathbf{1}_{(s \geq R)} \leq \left( s - \frac{s}{1+\delta R} \right) \mathbf{1}_{(s < R)} + s \mathbf{1}_{(s \geq R)}
\]

where \( \varepsilon_R(\delta) = 1 - \frac{1}{1+\delta R} \longrightarrow 0 \) as \( \delta \searrow 0 \). Hence, replacing \( s \) with \( f^n \) and splitting the integral as above, we get

\[
\|f^n - g^n_\delta\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} = \int \int |f^n \beta_\delta(f^n)| \, dx \, dv \leq \int \int f^n \varepsilon_R(\delta) \, dx \, dv + \int \int f^n \mathbf{1}_{(f^n \geq R)} \, dx \, dv
\]

\[
= \varepsilon_R(\delta) \int \int f^n_0 \, dx \, dv + \int \int f^n \mathbf{1}_{(f^n \geq R)} \, dx \, dv
\]

\[
\leq C \varepsilon_R(\delta) + \int \int f^n \mathbf{1}_{(f^n \geq R)} \, dx \, dv
\]

where \( C \) is a constant independent of \( n, R, \delta \), and it is finite because of the convergence (29). Now, since \( f^n \) is uniformly integrable in \( L^1((0,T) \times \mathbb{R}^d \times \mathbb{R}^d) \), we conclude

\[
\sup_n \sup_{t \in (0,T)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |g^n_\delta - f^n| \, dx \, dv \longrightarrow 0 \quad \text{as } \delta \searrow 0
\]

as claimed. \( \square \)

In view of (34), we focus on the time regularity of the functions \( g^n_\delta \). Observe that, since

\[
T g^n_\delta = \frac{1}{1+\delta f^n} Q^n(f^n, f^n)
\]

we have

\[
\frac{\partial g^n_\delta}{\partial t} = \left( \frac{1}{1+\delta f^n} Q^n(f^n, f^n) \right)^t
\]

and hence

\[
g^n_\delta(t + h) - g^n_\delta(t) = \int_t^{t+h} \frac{Q^n(f^n, f^n)^t(s)}{1+\delta f^n_\delta(s)} \, ds.
\]

Therefore for all \( t, h \) such that \( 0 < t < t + h < T \) we have

\[
\left\| g^n_\delta(t + h) - g^n_\delta(t) \right\|_{L^1(\mathbb{R}^d \times B_R)} = \int_{\mathbb{R}^d} \int_{B_R} \left| \int_t^{t+h} \frac{Q^n(f^n, f^n)^t(s)}{1+\delta f^n_\delta(s)} \, ds \right| \, dx \, dv
\]

\[
\leq \int_t^{t+h} \int_{\mathbb{R}^d} \int_{B_R} \frac{|Q^n(f^n, f^n)^t(s)|}{1+\delta f^n_\delta(s)} \, ds \, dx \, dv
\]

\[
= \int_t^{t+h} \int_{B_R} \int_{\mathbb{R}^d} \frac{|Q^n(f^n, f^n)|}{1+\delta f^n} \, ds \, dx \, dv
\]

Now, since the sequence \( \frac{Q^n(f^n, f^n)}{1+\delta f^n} \) is uniformly integrable in \( L^1((0,T) \times \mathbb{R}^d \times B_R) \), we can bound the right hand side by \( \phi(h) \), where \( \phi \) is a function which is independent of \( n \) and \( \delta \) such that \( \phi(h) \to 0 \) as \( \delta \searrow 0 \). Hence we deduce that

\[
\sup_n \sup_{t \in (0,T)} \left\| g^n_\delta(t + h) - g^n_\delta(t) \right\|_{L^1(\mathbb{R}^d \times B_R)} \to 0 \tag{35}
\]
as \(h \searrow 0\). Moreover,

\[
\sup_{t \in (0,T)} \|f^n(t + h) - f^n(t)\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} \leq \sup_{t \in (0,T)} \int \int |f^n(t + h) - f^n(t)| \, dx \, dv \\
\leq \sup_{t \in (0,T)} \left[ \int \int |f^n(t + h) - g^n(t + h)| + |g^n(t + h) - g^n(t)| (1_B + 1_{B_R}) + |f^n(t) - g^n(t)| \, dx \, dv \right] \\
\leq \phi'(h) + \sup_{t} \int \int |g^n(t + h) - g^n(t)|
\]

and by taking the supremum over \(n\) and using (35) we conclude that

\[
\sup \sup_{n \in (0,T)} \|f^n(t + h) - f^n(t)\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} \to 0
\]
as \(h \searrow 0\). Being the above convergence uniform in \(n\), the weak limit \(f^t\) of \(f^n\) also satisfies

\[
\sup_{t \in (0,T)} \|f^t(t + h) - f^t(t)\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} \to 0
\]
as \(h \searrow 0\), and therefore belongs to \(C((0,\infty), L^1(\mathbb{R}^d \times \mathbb{R}^d))\).

From this information we deduce that \(f \in C((0,\infty), L^1(\mathbb{R}^d \times \mathbb{R}^d))\) by the change of variables \((x + vh, v) \mapsto (x, v)\):

\[
0 \leftarrow \sup_{t \in (0,T)} \int \int |f^t(t + h) - f^t(t)| \, dx \, dv = \sup_{t \in (0,T)} \int \int |f(t + h, x + vh, v) - f(t, x + vh, v)| \, dx \, dv \\
= \sup_{t \in (0,T)} \int \int |f(t + h, x, v) - f(t, x, v)| \, dx \, dv = \sup_{t \in (0,T)} \|f(t + h) - f(t)\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)}
\]

and we conclude that

\[
f \in C((0,\infty), L^1(\mathbb{R}^d \times \mathbb{R}^d))
\]
as claimed.

### 3.6 Velocity averaging

To make use of the compactness results proved in the preceding section we go back to the sequences

\[
g^n_\delta = \frac{1}{\delta} \log(1 + \delta f^n)
\]
indexed by \(\delta > 0\), and recall that each \(g^n_\delta\) solves

\[
Tg^n_\delta = \frac{1}{1 + \delta f^n} Q^n(f^n, f^n).
\]  \hspace{1cm} (36)

The main advantage in using this sequences is that now we know that the RHS of (36) is weakly relatively compact in \(L^1((0,T) \times \mathbb{R}^d \times B_R)\) and such is, therefore, \(Tg^n_\delta\). Moreover, since

\[
0 \leq Tg^n_\delta \leq f^n
\]
and \(f^n\) is weakly compact in \(L^1((0,T) \times \mathbb{R}^d \times \mathbb{R}^d)\), we deduce from the DP criterion that \(Tg^n_\delta\) is also weakly compact in the same space. Therefore the velocity averaging results apply to \((g^n_\delta)\), and we deduce the following.

**Proposition 6.** For each \(T > 0\) we have:

(i) \(\int f^n \, dv \to \int f \, dv \) a.e. and in \(L^1((0,T) \times \mathbb{R}^d)\).

(ii) \(A_n \ast f^n \to A \ast f \) a.e. and in \(L^1((0,T) \times \mathbb{R}^d \times B_R)\), for all \(R > 0\).
(iii) For each function \( \varphi \) compactly supported in \( L^\infty((0,T) \times \mathbb{R}^d \times \mathbb{R}^d) \),
\[
\frac{\int Q^n_\pm f^n \varphi \, dv}{1 + \int f^n \, dv} \rightarrow \int \frac{Q_\pm f \varphi \, dv}{1 + \int f \, dv}
\]
as \( n \to \infty \) in \( L^1((0,T) \times \mathbb{R}^d) \).

Proof. Note that the family (indexed by \( \delta > 0 \)) of functions \( \beta_\delta(s) = \frac{1}{\delta} \log(1 + \delta s) \) satisfies all the assumptions of Lemma 5, namely \( (\beta_\delta) \) is a uniformly Lipschitz family in \( C(\mathbb{R}; \mathbb{R}) \) with \( \beta_0(0) = 0 \) and \( \beta_\delta(z) \to z \) as \( \delta \to 0 \) uniformly on compact sets. Moreover, as already noted, the sequence \( Tg^n_\delta = T\beta_\delta(f^n) \) is weakly relatively compact in \( L^1((0,T) \times \mathbb{R}^d \times B_R) \). Hence we can apply Corollary 1 with \( \psi_n \equiv \psi \equiv 1 \) to obtain the convergence of the velocity averages
\[
\int f^n \, dv \to \int f \, dv
\]
in \( L^1((0,T) \times \mathbb{R}^d) \). For (ii) we use the same truncation argument of step 2 in the proof of Theorem 5. By (12), which holds uniformly in \( n \), for any \( K > 0 \) we have
\[
\int_{|v_\ast| \leq R} A_n(v - v_\ast) \, dv_\ast = \left( \int_{|v_\ast| \leq R} A_n(v - v_\ast) \, dv_\ast \right) \left( 1_{|v| \leq K} + 1_{|v| > K} \right)
\]
\[
\leq \sup_{|v| \leq K} \left( \int_{|v_\ast| \leq R} A_n(v - v_\ast) \, dv_\ast \right) + C(1 + |v|^2) 1_{|v| > K}
\]
where the last integral is finite since \( A_n \in L^\infty_{loc}(\mathbb{R}^d; L^1(B_R)) \). We write \( A_{n,K}(v) := A_n(v - v_\ast) 1_{|v| > K} \). We have
\[
\| A_n * f^n - A * f \|_{L^1((0,T) \times \mathbb{R}^d \times \mathbb{R}^d; L^1(B_R))} \leq \| A_n * f^n - A_{n,K} * f^n \|_{L^1((0,T) \times \mathbb{R}^d \times \mathbb{R}^d; L^1(B_R))} + \| A_{n,K} * f^n - A_K * f^n \|_{L^1((0,T) \times \mathbb{R}^d \times \mathbb{R}^d; L^1(B_R))} + \| A_K * f^n - A * f \|_{L^1((0,T) \times \mathbb{R}^d \times \mathbb{R}^d; L^1(B_R))}.
\]
But by (12)
\[
\| A_n * f^n - A_{n,K} * f^n \|_{L^1((0,T) \times \mathbb{R}^d \times \mathbb{R}^d; L^1(B_R))} = \int_0^T \int_{|v| > K} A_n * f^n \, dv \, dx \, dt
\]
\[
\leq C \int_0^T \int |v| f^n(1 + |v|^2) \, dv \to 0
\]
as \( K \to \infty \) (recall that \( \sup_n \int f^n(1 + |v|^2) \, dv < \infty \)). So to prove the claim it is enough to show that
\[
\| A_{n,K} * f^n - A * f \|_{L^1((0,T) \times \mathbb{R}^d \times \mathbb{R}^d; L^1(B_R))} \to 0
\]
as \( n \to \infty \). To this end, define the sequence
\[
\phi^K_n(t, x, v, v_\ast) = A_{n,K}(v_\ast - v)
\]
and note that by (37) these functions are uniformly bounded in \( L^\infty((0,T) \times \mathbb{R}^d \times \mathbb{R}^d; L^1(B_R)) \). We can therefore apply Corollary 2 to get the strong convergence in \( L^\infty((0,T) \times \mathbb{R}^d \times B_R) \)
\[
\int \phi^K_n(t, x, v, v_\ast) f^n(t, x, v) \, dv \to \int \phi^K f \, dv = \int A_K(v_\ast - v) f(t, x, v) \, dv = A_K(t, x, v_\ast).
\]
By finally letting \( K \to \infty \) we obtain the strong convergence of \( A_n * f^n \) to \( A * f \) in \( L^\infty((0,T) \times \mathbb{R}^d \times B_R) \) as claimed.
To show the convergence (iii), fix any compactly supported function \( \varphi \in L^\infty((0,T) \times \mathbb{R}^d \times \mathbb{R}^d) \), and define the sequence
\[
\psi_n = \frac{A_n * f^n}{1 + \int f^n \, dv} \varphi.
\]
By (i) and (ii) $A_n * f^n$ and $(1 + \int f^n\,dv)$ converge almost everywhere to $A * f$ and $1 + \int f\,dv$ respectively, and hence
\[
\frac{A_n * f^n}{1 + \int f^n\,dv} \quad \to \quad \frac{A * f}{1 + \int f\,dv}
\quad \text{a.e.}
\]

We can therefore apply again Corollary 1 to get
\[
\frac{\int Q^n(f^n, f^n)\varphi\,dv}{1 + \int f^n\,dv} = \int f^n (A_n * f^n)\psi_n\,dv \quad \to \quad \int f(A * f)\varphi\,dv = \frac{\int Q(\cdot, f)\varphi\,dv}{1 + \int f\,dv}
\]
in $L^1((0, T) \times \mathbb{R}^d)$, and this concludes the proof.

\[\square\]

3.7 Exponential formulation

Another kind of solutions to the Boltzmann equation can be obtained by viewing $(A * f)\tilde{f}$ as a linear feedback term with variable coefficient $c(t, x, v) = A * f$ which we think of as a given function. The equation is then
\[
T\tilde{f} + c\tilde{f} = Q^+(\tilde{f}, f) = Q^+
\]
We then have the following form of solutions:
\[
\frac{d}{dt}\left[\exp\left(\int_0^t c^2\,ds\right)\tilde{f}\right] = \exp\left(\int_0^t c^2\,ds\right)Q^+\tilde{f}
\]
Duhamel’s principle then gives
\[
\exp\left(\int_0^t c^2\,ds\right)\tilde{f}^t = f_0 + \int_0^t \exp\left(\int_0^\tau c^2(s, x, v)\,ds\right)Q^+\tau\tilde{f}(\tau, x, v)\,d\tau
\]
This is unpleasant looking, but with some notation we can make it nicer.

Define $T^{-1}g$ as the solution to $T\tilde{f} = g$, $f|_{t=0} = 0$. Then $(T^{-1}g)^t = \int_0^t g^t(s, x, v)\,ds$. Also define $F = T^{-1}e$ so $F^t = \int_0^t c^2(s, x, v)\,ds$, then $T^{-1}_F = e^{-F}T^{-1}eF$ (an operator composition viewing $e^{\pm F}$ as multiplication operators). Then we see that
\[
\exp\left(\int_0^t c^2(t, x, v)\,ds\right) = e^{F^t(t, x, v)}
\]
\[
\int_0^t (e^F)^2Q^+\tau\tilde{f}(\tau, x, v)\,d\tau = (T^{-1}eFQ^+)\tilde{f} = (e^FT^{-1}_FQ^+)\tilde{f}
\]
So the equation is $(e^F)\tilde{f}^t = f_0 + (e^FT^{-1}_FQ^+)\tilde{f}^t$, and undoing the characteristic map $\tilde{f}$ and dividing by $e^F$ we obtain
\[
f = f_0e^{-F} + T^{-1}_F Q^+(f, f) \quad \text{where } F = T^{-1}(A * f)
\]
\[
e^{-F}(f_0 + T^{-1}[e^FQ^+(f, f)])
\]
This form has some advantages, in particular due to the explicit formula for $T^{-1}$ we can see that it is a continuous and weakly continuous map from $L^1((0, T) \times \mathbb{R}^d \times \mathbb{R}^d)$ to $C([0, T]; L^1(\mathbb{R}^d \times \mathbb{R}^d))$ which preserves non-negativity. Furthermore for $F \in C((0, T); L^1(\mathbb{R}^d \times \mathbb{R}^d))$, we see that $T^{-1}_F$ is strongly and weakly continuous from $L^1((0, T) \times \mathbb{R}^d \times \mathbb{R}^d)$ into $C((0, T); L^1(\mathbb{R}^d \times \mathbb{R}^d))$.

**Theorem 7.** We have
\[
f = f_0e^{-F} + T^{-1}_F Q^+(f, f)
\]
where $F = T^{-1}(A * f)$.

**Proof.** To prove this result, we first show that $f \geq f_0e^{-F} + T^{-1}_F Q^+(f, f)$. To this end, we once again consider the sequence $g^n_\delta = \frac{1}{\delta} \log(1 + \delta f^n)$. As shown earlier, this sequence is weakly (relatively) compact as $n \to \infty$, so we assume that
\[
g^n_\delta \to g_\delta
\]

for some $g_\delta$ weakly as $n \to \infty$. It is easy to see, as in the proof of Lemma \[5\] that \( \lim_{\delta \to 0} g_\delta = f \) as $\delta \to 0$ strongly in $L^1$. 

Similar arguments to the above tell us that the sequence

\[
l^n_\delta := \frac{f_n}{1 + \delta f_n}
\]

is weakly compact. Thus we may assume that $l^n_\delta \to l_\delta$ weakly as $n \to \infty$. Moreover, $l_\delta \to f$ as $\delta \to 0$ strongly in $L^1$.

We will also approximate $F$ by a sequence $F_n = T^{-1}(A_n * f_n)$ and we see that $T(e^{F_n}) = e^{F_n}(A_n * f_n)$ Finally, we notice that

\[
\frac{Q^n_+(f_n, f_n)}{1 + \delta f_n}
\]

is weakly compact, so we can assume

\[
\frac{Q^n_+(f_n, f_n)}{1 + \delta f_n} \to Q_{+,\delta}
\]

weakly for some $Q_{+,\delta}$.

Now that we have built up some convergent sequences, we perform the following computations

\[
T(e^{F_n}g^n_\delta) = e^{F_n}T(g^n_\delta) + g^n_\delta T(e^{F_n})
\]

\[
= e^{F_n}Q^n_+(f_n, f^n) + g^n_\delta e^{F_n}(A_n * f_n)
\]

\[
= e^{F_n}Q^n_+(f_n, f^n) + e^{F_n}(A_n * f_n) \left[ g^n_\delta - \frac{f_n}{1 + \delta f_n} \right]
\]

By integrating, we see that

\[
(e^{F_n}g^n_\delta)(t, x, v) - (e^{F_n}g^n_\delta)(0, x, v) = \int_0^t e^{F_n}Q^n_+(f_n, f^n)(s, x - (t - s)v, v)ds
\]

\[
+ \int_0^t e^{F_n}(A_n * f_n) \left[ g^n_\delta - \frac{f_n}{1 + \delta f_n} \right](s, x - (t - s)v, v)ds
\]

\[
= T^{-1} \left( e^{F_n}Q^n_+(f_n, f^n) \right)(t, x, v)
\]

\[
+ T^{-1} \left( e^{F_n}(A_n * f_n) \left[ g^n_\delta - \frac{f_n}{1 + \delta f_n} \right] \right)(t, x, v)
\]

If we define $T_{F_n}^{-1}$ by $T_{F_n}^{-1} : = e^{-F_n}T^{-1}e^{F_n}$ then the above equation reads

\[
g^n_\delta = \frac{1}{\delta} \log(1 + \delta f^n_\delta)e^{-F_n} + T_{F_n}^{-1} \left( (A_n * f_n) \left[ g^n_\delta - \frac{f_n}{1 + \delta f_n} \right] \right) + T_{F_n}^{-1} \left( Q^n_+(f_n, f^n) \right)
\]

(38)

We will need to bound the final term on the right hand side. To do this, we see that for any non-negative $\varphi \in L^\infty$ with compact support

\[
\int \left[ \frac{Q^n_+(f_n, f^n)}{1 + \delta f_n} \frac{\varphi}{1 + f_n dv_1} \right] dv \leq \int \left[ Q^n_+(f_n, f^n) \frac{\varphi}{1 + f_n dv_1} \right] dv
\]

From the prior results on velocity averaging Proposition \[6\] (ii), sending $n \to \infty$ yields

\[
\limsup_{n \to \infty} \int \left[ \frac{Q^n_+(f_n, f^n)}{1 + \delta f_n} \frac{\varphi}{1 + f_n dv_1} \right] dv \leq \int \left[ Q_+(f, f) \frac{\varphi}{1 + f dv_1} \right] dv
\]

But the left hand side converges to

\[
\int Q_{+,\delta} \frac{\varphi}{1 + f dv_1} dv
\]
and so
\[
\int Q_+ \varphi_1 + \frac{\varphi_1}{1 + \int f dv_1} dv_1 \leq \int Q_+ (f, f) \frac{\varphi_1}{1 + \int f dv_1} dv_1
\]
Because this holds for any \( \varphi \geq 0 \), we conclude that
\[
Q_+ \varphi_1 \leq Q_+ (f, f)
\]
almost surely in \( v \), and so because \( T^{-1} \) is a positive functional we have
\[
T^{-1}(Q_+ \varphi_1) \leq T^{-1}(Q_+ (f, f))
\]
It is easy to see that if \( x_n \to x \) weakly in \( L^1 \) then \( T_{F_n}^{-1}(x_n)(t) \to T_F^{-1}(x)(t) \) in \( L^1 \). We will now return to equation (38). Sending \( n \to \infty \) gives
\[
g_\delta = \frac{1}{\delta} \log(1 + \delta f_0) e^{-F} + T_F^{-1} ((A * f)[g_\delta - l_\delta]) + T_F^{-1}(Q_+ \varphi_1)
\]
\[
\leq \frac{1}{\delta} \log(1 + \delta f_0) e^{-F} + T_F^{-1} ((A * f)[g_\delta - l_\delta]) + T_F^{-1}(Q_+ (f, f))
\]
Notice that
\[
\frac{1}{\delta} \log(1 + \delta f_0) = \log \left[ (1 + \delta f_0)^{\frac{1}{\delta}} \right] \to \log(e^{f_0}) = f_0
\]
as \( \delta \to 0 \).
Also by the continuity of \( T_F^{-1} \),
\[
T_F^{-1}((A * f)[g_\delta - l_\delta]) \to T_F^{-1}((A * f)[f - f]) = 0
\]
and so
\[
f \leq f_0 e^{-F} + T_F^{-1} Q_+ (f, f)
\]
It remains to show that
\[
f \geq f_0 e^{-F} + T_F^{-1} Q_+ (f, f)
\]
Our argument to show this will be similar to the one we have just performed. This time, we consider the sequence of cut-off functions \( (h^n_\delta) \) such that \( h^n_\delta = \min(f^n, \frac{1}{\delta}) \). It is clear that \( h^n_\delta \to h_\delta \) weakly for some \( h_\delta \) satisfying
\[
h_\delta \uparrow f
\]
We also get
\[
Q_n^+ (h^n_\delta, h^n_\delta) \to Q_+ (h_\delta, h_\delta)
\]
weakly. A similar computation to the one earlier (except this time considering \( T(e^F f_n) \)) gives
\[
f^n = f_0^n e^{-F} + T_{F_n}^{-1}(Q_n^+ (f^n, f^n))
\]
But \( f^n \geq h^n_\delta \), and again because \( T_{F_n}^{-1} \) is positive
\[
f^n \geq f_0^n e^{-F} + T_{F_n}^{-1}(Q_n^+ (h^n_\delta, h^n_\delta))
\]
We send \( n \to \infty \) in this and conclude that
\[
f \geq f_0 e^{-F} + T_F^{-1}(Q_+ (h_\delta, h_\delta))
\]
Next, by the monotone convergence theorem,
\[
Q_+ (h_\delta, h_\delta) \uparrow Q_+ (f, f)
\]
and so
\[
f \geq f_0 e^{-F} + T_F^{-1}(Q_+ (f, f))
\]
as was claimed.

**Theorem 8.** The \( f \) constructed above is indeed a renormalized solution.
Proof. Firstly, we have
\[
0 \leq \frac{Q_-(f, f)}{1 + f} = \frac{(A * f)f}{1 + f} \leq A * f
\]
But we know from Lemma 6 that \( \lim_{n \to \infty} A_n * f^n \to A * f \) in \( L^1((0, T) \times \mathbb{R}^d \times B_R) \) sense for all \( T, R > 0 \). Thus
\[
\frac{Q_-(f, f)}{1 + f} \in L^1((0, T) \times \mathbb{R}^d \times \mathbb{R}^d_{loc})
\]
for any \( T > 0 \).

We will show the (harder) corresponding result for \( \frac{Q_+(f, f)}{1 + f} \). To do this, recall that, from the entropy inequality [13],
\[
Q_+^n(f^n, f^n) \leq 2Q_+^n(f^n, f^n) + 4e_n(f^n)\frac{e_n(f^n)}{\log(2)}
\]
Next, for any positive \( \delta \), we divide by the positive function \( (1 + \delta \int f^n dv) \) to get
\[
\frac{Q_+^n(f^n, f^n)}{1 + \delta \int f^n dv} \leq 2\frac{Q_+^n(f^n, f^n)}{1 + \delta \int f^n dv} + 4\frac{e_n(f^n)}{\log(2)}(1 + \delta \int f^n dv)
\]
We have shown in Proposition 6 that \( \int f^n dv \to \int fdv \). We also showed in that same proposition that for any function \( \varphi \in L^\infty((0, T) \times \mathbb{R}^d \times \mathbb{R}^d) \) with compact support
\[
\int_{\mathbb{R}^d} Q_+^n(f^n, f^n) \varphi dv \to \int_{\mathbb{R}^d} Q_+(f, f) \varphi dv
\]
Moreover, since \( e_n(f^n) \) are \( L^1(\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d) \) uniformly bounded, they converge vaguely to some finite measure \( \mu \). The absolutely continuous part of \( \mu \) (which we denote by \( e \)) is in \( L^1((0, T) \times \mathbb{R}^d \times \mathbb{R}^d) \) and is non-negative and so, by taking limits in \( [39] \)
\[
\int_{\mathbb{R}^d} Q_+(f, f) \varphi dv \leq \int_{\mathbb{R}^d} Q_+(f, f) \varphi dv + 4\frac{e(\varphi)}{\log(2)}
\]
By sending \( \delta \to 0 \), we see that
\[
\int_{\mathbb{R}^d} Q_+(f, f) \varphi dv \leq \int_{\mathbb{R}^d} Q_+(f, f) \varphi dv + 4\frac{e(\varphi)}{\log(2)}
\]
and because \( e \) is absolutely continuous we get
\[
Q_+(f, f) \leq Q_+(f, f) + E
\]
where \( E \in L^1((0, T) \times \mathbb{R}^d \times \mathbb{R}^d) \) is non-negative. Thus
\[
\frac{Q_+(f, f)}{1 + f} \leq \frac{Q_-(f, f)}{1 + f} + E\frac{1}{1 + f} \leq \frac{Q_-(f, f)}{1 + f} + E
\]
and so \( \frac{Q_+(f, f)}{1 + f} \in L^1((0, T) \times \mathbb{R}^d \times \mathbb{R}^d_{loc}) \).

Next, we show that \( Q_+(f, f)^h(t, x, v) \) is in \( L^1(0, T) \) for almost all \((x, v)\). To do this, recall that by Lemma 7 we have \( T_{1}^{-1}(Q_+(f, f)^h(t, x, v)) \in L^1(\mathbb{R}^d \times \mathbb{R}^d_{loc}) \) for every \( t > 0 \) and so \( [T_{1}^{-1}(Q_+(f, f)^h(t, x, v))]^\# \in L^1(\mathbb{R}^d \times \mathbb{R}^d_{loc}) \). Writing this out fully, we see that for compact sets \( K \subset \mathbb{R}^d \)
\[
\begin{align*}
\infty > & \int_{K} \int_{\mathbb{R}^d} (e^{-F(t)})^h(x, v) \left[T_{1}^{-1}(Q_+(f, f)^h(t, x, v))\right]^\# dx dv \\
> & \int_{K} \int_{\mathbb{R}^d} (e^{-F(t)})^h(x, v) \int_{0}^{t} e^{F^\#(s)} \left(Q_+(f, f)^h(s, x, v)\right) ds dx dv \\
> & \int_{K} \int_{\mathbb{R}^d} \int_{0}^{t} e^{-(F^\#(s)-F^\#(t))}(x, v) \left(Q_+(f, f)^h(s, x, v)\right) ds dv dx \\
\end{align*}
\]
But $F^\#(t, x, v) = \int_0^t (A * f)(s, x, v)$. We know that $A * f$ is non-negative, so $F^\#$ increases in $t$ and is also non-negative. Thus

$$\int_R \int_{R^d} \int_0^t e^{-(F^\#(t))(x, v)} (Q^\#_+(f, f))(s, x, v) ds dxdv$$

It is clear that for each $t > 0$ we have $F^\#(t) \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$. Define $\Omega_0$ to be the set of $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$ such that $F^\#(t) < \infty$. Then $\Omega_0^c$ is null.

We also define $\Omega_1$ to be the set of $(x, v)$ such that $\int_0^t e^{-(F^\#(t))(x, v)} (Q^\#_+(f, f))(s, x, v) ds < \infty$.

By (41), $\Omega_1^c$ is also null. On $\Omega_0 \cap \Omega_1$, which has null complement, we have

$e^{-F^\#(t)} > 0$

Thus, on this set

$$\int_0^t (Q^\#_+(f, f))(s, x, v) ds < \infty$$

which proves that $Q_+(f, f)^\#(t, x, v)$ is in $L^1(0, T)$ for almost all $(x, v)$. By [40], $Q_-(f, f)^\#(t, x, v)$ is in $L^1(0, T)$ for almost all $(x, v)$. Then by Theorem 1, $f$ is a renormalized solution.

4 Rough coefficients

4.1 Another use for renormalised solutions

In the previous section we saw how the notion of renormalised solutions allowed a quadratic non-linearity to be controlled. Another use of renormalised solutions is to make sense of very irregular solutions. In fact, we can have solutions that are only measurable and finite a.e. in the space $L^1$. In this space distributional solutions do not make sense. We will give a different definition of a solution in such a space. To do this, instead of using the derivative of $\beta$ to bring down a non-linearity, we will use bounded $\beta$ to control the spikes of the solution. We will show that even in this space we have the stability of solutions subject to perturbations of the coefficients, which allow us to gain knowledge about the characteristic curves even when standard ODE theory would not tell us about their existence. In particular, we will state a generalised version of Cauchy-Lipschitz that applies even when the force-field term is not Lipschitz. This result is due to [3], and an adapted version of the time independent divergence free case is presented here.

4.1.1 Cauchy-Lipschitz with rough coefficients

The well known Cauchy-Lipschitz\footnote{Also called Picard-Lindelof} theorem for ODEs gives existence and uniqueness for locally Lipschitz forcefields. We will obtain a slightly weaker result under the assumption that the force-fields are merely $W^{1,1}_{loc}$, i.e. almost Lipschitz. The solutions will exist for only almost all initial data.

**Theorem 9** (Main result). Let $b : \mathbb{R}^N \to \mathbb{R}^N$, $b \in W^{1,1}_{loc}$, $\text{div} \ b = 0$ a.e., and $\frac{b}{1+|x|} \in L^1 + L^\infty$. Then the equation

$$\begin{align*}
\dot{X} &= b(X) \\
X(0) &= x
\end{align*}$$

has a unique solution $X(t, x)$ in the sense that for almost all $x \in \mathbb{R}^N$,

\footnote{i.e. $\frac{b}{1+|x|} = b_1 + b_2$, $b_1 \in L^1$, $b_2 \in L^\infty$. See section 2 for more details.}
• The equation is satisfied for $t \in \mathbb{R}$.
• $X(\cdot, x) \in C^1(\mathbb{R})$, $b(X) \in C(\mathbb{R})$.

4.1.2 Strategy

The Cauchy-Lipschitz theorem is usually proven using a fixed point argument involving the Lipschitz condition. Here we will use a completely different method; we will view Cauchy-Lipschitz as a statement about PDEs.

**Theorem 10** (Cauchy-Lipschitz (smooth case)). Let $b \in C^\infty(\mathbb{R}^N; \mathbb{R}^N)$. Then the ODE stated in (42) has a unique smooth solution (in both variables) $X(t, x)$ and

$$\frac{\partial X_i(t, x)}{\partial t} = b_i(X)$$

In particular, the solutions of the ODE are the characteristics of the transport equation

$$\frac{\partial u}{\partial t} - b_i \frac{\partial u}{\partial x_i} = 0$$

with $u(0, x) = x$.

Instead of looking at the original ODE we will attack the transport equation. We will derive stability results using renormalised solutions, and then return to the original ODE via the characteristics.

4.1.3 Techniques and ideas used

Other than the notion of renormalised solutions, the proofs that follow use standard regularising and approximating methods.

4.2 A proof using the stability result

We now have the machinery to state the stability result that allows us to prove the main theorem.

**Theorem 11** (Stability). Let $b^n \to b$, $c^n \to c$, $\text{div} \ b^n \to \text{div} \ b$ respectively in $L^1_{\text{loc}}$ and assume that $b$, $\text{div} \ b$ and $c$ satisfy

$$b \in W^{1,1}_{\text{loc}}, \text{div} \ b \in L^\infty, c \in L^\infty$$

(44)

$$\frac{|b|}{1 + |x|} \in L^1 + L^\infty$$

(45)

Furthermore, let $u^n$ be the renormalized solution to (1) (with $b$ replaced by $b^n$ etc.) with initial data $u^n_0 \in L$ such that $u^n_0$ converges in $L$ to some $u_0 \in L$. We also assume that $u^n$ is a bounded sequence in $L^\infty(L)$. Then $u^n \to u$ in $C(L)$ where $u$ is the renormalized solution of the original transport equation (1).

**Theorem 12** (Existence and measure preservation). Let $u_0 \in L$. Then there is a unique renormalized solution $u$ to (1) such that $u \in L^\infty(L) \cap C(L)$.

We shall now restate and prove the main theorem using these results. Later, we will show that these results are indeed valid.

**Theorem 13** (Main result). Assume that (44) and (45) are satisfied and that $c$, $\text{div} \ b = 0$ and $b$ depend only on $x$. Then there is a unique $X(t, x) \in C(\mathbb{R}; L)$ such that a.e. in $x$ we have

(i) $X(\cdot, x) \in C^1(\mathbb{R})$, $b(X(\cdot, x)) \in C(\mathbb{R})$

(ii) $\frac{\partial X_i}{\partial t} = b_i(X)$ i.e. the original ODE.

Also (with no a.e. restriction) we have
(a) For a fixed \( t \), \( X(t,x) \) preserves Lebesgue measure (i.e. \( \mu \circ X(t,\cdot) = \mu \) where \( \mu \) is the Lebesgue measure) so that for all test functions \( \phi \) we have \( \int \phi(X(t,x))dx = \int \phi(x)dx \).

(b) (Group property) For all \( t,s \) and for almost all \( x \) we have \( X(t+s,x) = X(t,X(s,x)) \).

(c) The equation
\[
\frac{\partial \beta_j(X)}{\partial t} = \sum_{i=1}^d \frac{\partial b_j(x)}{\partial x_i} \beta_i(X)
\]  
(46)
holds in the sense of distributions for all \( \beta \in C^1(\mathbb{R}^N;\mathbb{R}^N) \) with \( \beta, (D\beta)(z)(1+|z|) \in L^\infty \). This is the notion of renormalized solutions restated in the \( \mathbb{R}^N \rightarrow \mathbb{R}^N \) case.

(d) \( \beta(X) \in L^1_{\text{loc}}(\mathbb{R}^N;C(\mathbb{R})) \) for all such \( \beta \).

Proof. We start with existence. The plan is as follows:

1. Mollify \( b \) and use the smooth version of Cauchy-Lipschitz and characteristics to formulate a sequence of solutions to the transport equation.
2. Fix a \( \beta \), show compactness and apply the stability result to gain existence and uniqueness of the solution to (46).
3. Deduce measure preservation and group property from the transport equation.
4. Use properties of a particular \( \beta \) to deduce continuity.
5. Use similar compactness ideas to deduce that \( b(X) \) is continuous.
6. Use (46) to deduce the statements that hold for almost all \( x \).

We will now execute this plan:

1. Apply the smooth version of Cauchy-Lipschitz to solve \( \dot{X} = \tilde{b}(\dot{X}) \) where \( \tilde{X} \) and \( \tilde{b} \) are obtained by a convolution with the function \( \frac{1}{\epsilon^N} \rho(\cdot/\epsilon) \) where \( \rho \in \mathcal{D}(\mathbb{R}^N) \) and \( \int \rho = 1 \). Doing this yields a unique smooth \( \tilde{X}(t,x) \) which obeys \( \frac{\partial \tilde{X}}{\partial t} = \tilde{b}(\tilde{X}) \) and \( \tilde{X}(0,x) = x \). Furthermore, provided \( u_0 \) is smooth, we can use the chain rule, the group property and the measure preservation of \( \tilde{X} \) to see that \( \tilde{u} = u^0(\tilde{X}) \) solves
\[
\frac{\partial \tilde{u}}{\partial t} = \tilde{b} \frac{\partial \tilde{u}}{\partial x_i}, \quad \tilde{u}(0,x) = u_0(x)
\]
and so by choosing \( u^0(x) = x_i \) (i.e. \( u^0 \) is a projection) we deduce that \( \tilde{X} \) satisfies
\[
\frac{\partial \tilde{X}}{\partial t} = \tilde{b} \frac{\partial \tilde{X}}{\partial x_i}, \quad \tilde{X}(0,x) = x
\]

2. As everything is smooth, we know that a function is a renormalised solution to the perturbed ODE if and only if it is a solution to the perturbed ODE. We wish to extract a subsequence from \( \tilde{X} \) that converges for any \( \beta \) that obeys the conditions on (46). By combining the equations \( \frac{\partial \beta_j}{\partial t} = \dot{b}_j(\tilde{X}) = \sum_i \frac{\partial b_j(x)}{\partial x_i} \beta_i(\tilde{X}) \) and the chain rule we see that
\[
\frac{\partial \beta_j(X)}{\partial t} = \sum_i \frac{\partial b_j}{\partial x_i} \beta_i(\tilde{X}) = \sum_i \frac{\partial \beta_i}{\partial x_i} \frac{\partial b_j}{\partial x_i} \tilde{X}
\]
which is the perturbed version of (46). Our assumptions on \( \beta \) and (45) tell us that for every \( \dot{i} \), \( \frac{\partial \beta_i(\tilde{z})}{\partial x_j} \tilde{b}_j(\tilde{z}) \in L^1 + L^\infty \). By the measure preservation property this is true for \( z = \tilde{X} \), i.e. \( \frac{\partial \beta_i}{\partial x_j} \tilde{b}_j(\tilde{X}) \in L^1 + L^\infty \) uniformly in \( t \), and we obtain

\footnote{Here and throughout, \( \frac{\partial \beta_i}{\partial x_j} \) means \( \frac{\partial \beta_i}{\partial x_j} \) evaluated at \( \tilde{X} \).}
\[ \frac{\partial [\beta_i(\tilde{X})]}{\partial t} \text{ is bounded in } L^\infty(\mathbb{R}; L^1 + L^\infty) \]

\[ \frac{\partial [\beta_i(\tilde{X})]}{\partial x} \text{ is relatively compact in the sets } L^\infty(-T, T; L^1(B_R)) \text{ for all } R, T < \infty. \]

This tells us that \( \beta(\tilde{X}) \) bounded in \( L^\infty(\mathbb{R}; L^1_{loc}) \). This is enough to apply the stability result, hence \( \tilde{X} \) converges in \( C([-T, T]; L^N) \) for all \( 0 < T < \infty \) to an \( X \) that solves the original \( \text{(46)} \).

3. We can use these \( \beta \) to approximate the test functions and obtain (a) using the measure preservation result on the transport equation. The group property follows from the time invariance of the transport equation. We observe that \( X(t, X(s, x)) \) solves \( \frac{\partial u}{\partial s} - b_i \frac{\partial u}{\partial x_i} = 0 \) for \( u(0, x) = X(t, x) \) in the sense of a renormalised solution.

4. We repeat the reasoning given in step 2 on equation \( \text{(46)} \) to obtain that \( \frac{\partial [\beta_i(\tilde{X})]}{\partial t} \) is continuous. Fixing any compact interval and choosing a \( \beta \) such that \( \beta(\tilde{X}) \) is strictly increasing on this interval tells us that \( X \) is continuous on that interval, and hence \( X \) is continuous on all of \( \mathbb{R} \).

5. Doing the same for \( b(\tilde{X}) \) is slightly trickier. We introduce a positive even \( C^1(\mathbb{R}) \) function \( \psi \) and look at

\[ \frac{\partial}{\partial t} \left[ \psi(\tilde{X}) \beta_i(\tilde{b}(\tilde{X})) \right] = \frac{\partial \psi}{\partial \tilde{X}_j} \frac{\partial \tilde{X}_j}{\partial t} \beta_i(\tilde{b}(\tilde{X})) + \psi(\tilde{X}) \frac{\partial \beta_i}{\partial \tilde{b}_j} \frac{\partial \tilde{b}_j}{\partial \tilde{X}_k} \frac{\partial \tilde{X}_k}{\partial t} \]

By a Sobolev embeddings argument and the fact that \( Db \in L^1_{loc} \) we know that \( b \in L^N_{loc} \), because log grows sufficiently slowly. We can now pick \( \psi \) so that:

\[ \psi(|z|)|Db(z)| \leq f(z) \in L^1_+ \]

\[ \psi'(|z|)|b| \log(1 + |b|) \leq g(z) \in L^1_+ \]

for some \( f, g \) and uniformly for \( \epsilon \in [0, 1] \). We can use this to bound the time derivative as follows:

\[ \left| \frac{\partial}{\partial t} \left[ \psi(\tilde{X}) \beta_i(\tilde{b}(\tilde{X})) \right] \right| \leq g(\tilde{X}) \| \beta \|_\infty + f(\tilde{X}) |zD\beta(z)| \]

\[ \leq Cg(\tilde{X}) + f(\tilde{X}) \| D\beta(z) (1 + |z|) \|_\infty \]

\[ \leq Cg(\tilde{X}) + C'f(\tilde{X}) \in L^1_+ \]

Because \( \tilde{X} \) preserves measure we have that \( \frac{\partial}{\partial t} \left[ \psi(\tilde{X}) \beta_i(\tilde{b}(\tilde{X})) \right] \) is bounded in \( L^\infty(\mathbb{R}; L^1) \) and weakly relatively compact in \( L^1(-T, T; L^1(B_R)) \) for all finite \( R, T \). We then send \( \epsilon \to 0 \) and obtain \( \frac{\partial}{\partial t} \left[ \psi(X) \beta_i(b(X)) \right] \in L^\infty(\mathbb{R}; L^1) \), and so by using the same method as before, for almost all \( x \in \mathbb{R}^N \), \( \psi(X) \beta(b(X)) \) is continuous on \( \mathbb{R} \). Additionally, for almost all \( x \), \( X \in C(\mathbb{R}, \psi \in C^1(\mathbb{R}) \) and so we can choose \( \beta \) to see that \( b(X) \) is continuous on \( \mathbb{R} \).

6. Now we just let \( \beta \) tend to the identity, and use our almost everywhere continuity of \( X \) and \( b(X) \) to deduce the integral form of \( \tilde{X} = b(X) \). This gives the almost everywhere existence of solutions to the originally ordinary differential equation.

Now we will show uniqueness. It is sufficient to show that for arbitrary \( u^0 \in \mathcal{D}(\mathbb{R}^N) \), \( u(t, x) = u^0(X(t, x)) \) solves the transport equation with \( c = 0 \) (which we already know has a unique solution). Essentially we are showing that \( X \) is the characteristic map for the transport equation with a test function as initial data.

We have already proved the existence of an \( X \) such that for almost all \( x \), \( X(\cdot, x) \in C^1(\mathbb{R}) \) and \( X \) satisfies the ODE. As \( u^0 \) is smooth, we can apply the chain rule to obtain that for almost all \( x \),

\[ \frac{du}{dt} = \frac{\partial u}{\partial X_i} b_i(X) \]

38
But this is just the statement that \( u \) is a mild solution to \( \frac{du}{dt} - b_i \frac{\partial u}{\partial x_i} = 0 \), and certainly \( u \in L^1_{loc}(\mathbb{R} \times \mathbb{R}^N) \) as it has compact support. The forcing term is \( 0 \in L^1_{loc}(\mathbb{R} \times \mathbb{R}^N) \). Also \( X \) is invertible by the group property and preserves measure, so we can apply theorem 1 to show that \( u \) is a distributional solution and hence unique.

\[ \square \]

4.3 Proof of stability result

We have already seen that the transport equation has some stability in lemma 2. However, the most general result requires many separate approximation arguments, so to simplify things we will only consider the case where \( c, c_n = 0 \) and \( b \) depends only on \( x \) (which was all that was needed above). We will look at the equation

\[ \frac{\partial u}{\partial t} - b_i \frac{\partial u}{\partial x_i} = 0 \]

with \( b^n \to b \) and \( \text{div} b^n \to \text{div} b \) in \( L^1_{loc} \). To remove some of the sub/super-scripts we will no longer assume that \( \tilde{u} \) is a convolution with \( \rho \) as in the previous section: instead, we will write \( \tilde{u} = u^n \) and adopt a similar notation for other functions. The sequence of initial conditions will then be given by \( \tilde{v} \in L^1_{loc} \). Hence \( \tilde{v} \rightarrow \tilde{v}^0 \). Without loss of generality we can assume that \( v^n \) converge in \( L^\infty((0, T) \times \mathbb{R}^N) \) to some \( v \) in the weak* sense. By the convergence of \( \tilde{b} \) and \( \text{div} \tilde{b}, v \) will solve the equation, and because of the convergence of \( \tilde{u} \) it has initial condition \( v^0 = \beta(u^0) \).

In the same way we can take \( \tilde{w} = \beta(u)^2 \), to obtain a \( w \) with \( \tilde{w} \to w \) such that \( w \) solves the equation with initial condition \( w^0 = \beta(u^0)^2 \). But then because \( v \) solves the equation it is a renormalised solution by theorem 3. Hence \( v^2 \) (the square of \( v \)) solves the equation with initial condition \( \beta(u^0)^2 = w^0 \). By uniqueness we then have that \( v^2 = w \). However, \( v^2 \to v^2 \) weak* in \( L^\infty((0, T) \times \mathbb{R}^N) \) and hence in \( L^2(L^2_{loc}) \). This gives us convergence locally in measure. As \( \tilde{v} = \beta(\tilde{u}) \) where \( \beta \) was arbitrary (under the conditions imposed), we can vary \( \beta \) over strictly increasing functions to deduce that \( \tilde{u} \to u \) locally in measure to some \( u \). Therefore \( \tilde{v} \to \beta(u) \) and so \( u \) is a renormalised solution.

4.3.3 On compact time sets

We now need to show that the convergence is uniform on our time interval \([0, T] \). To begin with we will show that for all such \( \beta \) we have \( \beta(\tilde{u}) \to \beta(u) \) in \( C(0, T; L^1_{loc}) \). It suffices to prove that if \( \tilde{t} \to t \) in \([0, T] \) then \( \beta(\tilde{u}(\tilde{t})) \to \beta(u(t)) \) in \( L^2(B_R) \) for all balls \( B_R \). Because \( B_R \) is of finite measure, it is sufficient to show \( \beta(\tilde{u}(\tilde{t})) \to \beta(u(t)) \) in \( L^2(B_R) \). Applying our equation with a test cut off function \( \phi_R \) we have

\[ \frac{d}{dt} \int \tilde{w}\phi_R \, dx = \int \tilde{b}_i \frac{\partial \tilde{w}}{\partial x_i} \phi_R + \tilde{w} \tilde{b}_i \frac{\partial \phi_R}{\partial x_i} \, dx \]

The convergence of \( \tilde{b} \) and \( \tilde{w} \) as \( n \to \infty \) in \( L^1_{loc} \) then give

\[ \frac{d}{dt} \int \tilde{w}\phi_R \, dx \to \int \tilde{b}_i \frac{\partial \tilde{w}}{\partial x_i} w\phi_R + \tilde{w} \tilde{b}_i \frac{\partial \phi_R}{\partial x_i} \, dx \]

in \( L^1(0, T) \).
Hence $\int \tilde{v}^2 \phi_R \, dx \to \int v^2 \phi_R \, dx$ uniformly in $[0, T]$, and $\int \tilde{v}(t)^2 \phi_R \, dx \to \int v(t)^2 \phi_R \, dx$ in $C[0, T]$. Now we consider $s$ sufficiently large so that $(\tilde{v})$ is relatively compact in $H^{-s}(B_R)$. Then $\tilde{v}(t) \to v(t)$ in $H^{-s}(B_R)$ and hence weakly in $L^2(B_R)$. Combining this with the uniform convergence of integrals above we obtain that $\tilde{v}(t) \to v(t)$ in $L^2(B_R)$, and because $v \in C(0, T; L^2(B_R))$ we have that $\tilde{v} \to v$ in $C(0, T; L^2(B_R))$.

Now we have to show that this is enough to deduce that $\tilde{u} \to u$ in $C(0, T; L)$. For uniformly bounded $\tilde{u}$ and $u$ we can use $\beta(z) = |z| \wedge M$ to obtain local convergence in measure. It is therefore enough to show that we can reduce the unbounded case to this. To do this we note that for any measurable $f : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ that is finite a.e., we have that for any fixed ball $B_R \subset \mathbb{R}^N$,

$$\sup_{t \in [0, T]} \mu(\{|\beta(t, \cdot)| \geq M\} \cap B_R) \leq \mu \left( \left\{ \sup_{t \in [0, T]} |\beta(t, \cdot)| \leq M \right\} \cap B_R \right)$$

As $g(x) := \sup_{t \in [0, T]} |\beta(t, x)|$ is measurable and finite a.e. we have that $1 \geq 1 \{|g| \geq M\} \to 0$. By the dominated convergence theorem on the bounded domain $B_R$ we see that

$$\sup_{t \in [0, T]} \mu(\{|\beta(t, \cdot)| \geq M\} \cap B_R) \to 0$$

We choose an $M$ sufficiently large to make the this less than $\epsilon/2$ uniformly over $f = \tilde{u}$ and use the bounded case to obtain a total error that is bounded above by $\epsilon$.

4.3.4 Existence and uniqueness

Existence and uniqueness are now easy. We just use mollify the coefficients and initial conditions with parameters $\epsilon$ and $\delta$ respectively, and then solve the equation to find $\beta(\tilde{u})$ which is unique by standard results. We choose $\beta$ to be as in the definition of renormalized solutions, with $\beta_0$ strictly increasing. Taking $\delta \to 0$ we get existence and uniqueness of $\tilde{u}$ as a renormalised solution to the equation with mollified coefficients. The stability result then allows us to send $\epsilon \to 0$.

4.4 Further remarks

In [3] the bounded divergence and non-autonomous cases are considered. The result is similar to what is stated here, but the group property fails, and measure preservation is no longer exact, becoming exponential upper and lower bounds based on the bounds on the divergence. They also prove versions of the stability result when $c \neq 0$ the coefficients are time-dependent, and for initial conditions in $L^p$ spaces. The methods of proof are similar, and so we omitted them here for the sake of brevity. The results for ODEs are also shown to be sharp, in the sense that there is non-uniqueness of solutions to the ODE for some divergence free $b \in W^{s,1}_{loc}$ for any $0 \leq s < 1$, and also for some time-independent $b \in W^{1,p}_{loc}$ for any $p$ without any divergence assumption.

5 Conclusions

In this project, we have seen how the idea of renormalised solutions has allowed us to provide a solution to the Boltzmann equation. We have derived various stability results to help us achieve this. Next, we explored applications of these techniques to transport equations with rough coefficients. These surprising results were of great interest to us and we hope that the reader found this project to be informative and enjoyable. We briefly mentioned some of the many interesting remaining open problems in this field. The various difficulties encountered in this project should suggest why these interesting problems remain unsolved despite plenty of attention.

A Appendix

A.1 Weak* compactness

**Theorem 14** (Banach-Alaoglu). Let $X$ be a normed vector space. Then the closed unit ball of $X^*$ is weak* compact.

**Corollary 4.** A set is relatively weak* compact if and only if it is bounded.
A.2 Weak compactness

**Definition 18** (Uniformly integrable). A set $F \subset L^1(U)$ is uniformly integrable if and only if

(a) $F$ is bounded in $L^1(U)$.

(b) For any $\epsilon > 0$ there is a $\delta > 0$ such that

$$\int_A |f| < \epsilon \quad \text{for every } f \in F \text{ and for every } A \subset U \text{ with } \int_A 1 < \delta$$

(c) $F$ is tight when we view $f \in F$ as the measure $|f|\mu$. In other words, for every $\epsilon > 0$ there is a compact $K \subset U$ with $\int_{U\setminus K} |f| \leq \epsilon$.

**Theorem 15** (Dunford-Pettis). A collection of functions is relatively weakly compact in $L^1$ if and only if it is uniformly integrable.

**Corollary 5.** Suppose that $|f_n| \leq |g_n|$ where $(f_n), (g_n) \subset L^1$ and $(g_n)$ is relatively weakly compact. Then $(f_n)$ is relatively weakly compact.

**Proof.** By Dunford Pettis, $(g_n)$ is uniformly integrable. Fix $\epsilon > 0$. Then we see that as $|f_n| < |g_n|$, the same $\delta$ that satisfies $[\text{b}]$ in the definition of uniform continuity for $(g_n)$ will also apply for $(f_n)$. Hence $(f_n)$ is uniformly integrable, and relatively weakly compact. \hfill \Box

**Lemma 15** (Weak compactness in $L^p$). Let $p \in (1, \infty)$ and $A \subset L^p$ be bounded. Then $A$ is relatively weakly compact.

A.3 Strong compactness

**Theorem 16** (Rellich-Kondrachov (special case)). Let $U \subset \mathbb{R}^N$ be open bounded and Lipschitz, and $s < t$. Then $H^s(U) \subset \subset H^t(U)$.

A.4 Continuity in Banach spaces

**Lemma 16.** Let $X$ be a real Banach space. Suppose the function $u : [0, T) \times X \to \mathbb{R}$ satisfies $\frac{\partial u}{\partial t} \in X$ and $u \in X$ for a.a. fixed $t$. Then $u(t) \in C((0, T]; X)$.

References

